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INTUITIONISTIC LOGIC IN INTUITIONISTIC METAMATHEMATICS

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PROMOTOR:

PROF. J.J. DE IONGH

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C O N T E N T S

I N T R O D U C T I O N

Let IPC be the intuitionistic first-order predicate calculus. From the definition of derivability in IPC the following is clear:

(1) If A is derivable in IPC, denoted by " $\vdash_{IPC} A$", then A is intuitively true, that means, true according to the intuitionistic interpretation of the logical symbols. To be able to settle the converse question: "if A is intuitively true, then $\vdash_{IPC} A$", one should make the notion of intuitionistic truth more easily amenable to mathematical treatment. So we have to look then for a definition of "A is valid", denoted by " $\models A$", such that the following holds:

(2) If A is intuitively true, then $\models A$.
And then one might hope to be able to prove

(3) If $\models A$, then $\vdash_{IPC} A$.

If one would succeed in finding a notion of " $\models A$", such that all the conditions (1), (2) and (3) are satisfied, then the chain would be closed, i.e. all the arrows in the scheme below would hold.

$$\begin{tikzcd}
\vdash_{IPC} A \arrow[rr] & & A \text{ is intuitively true} \arrow[dl] \\
& \models A \arrow[ul] &
\end{tikzcd}$$

Several suggestions for $\models A$ have been made in the past:
Topological and algebraic interpretations, see Rasiowa and Sikorski [1].
The intuitionistic models of Beth, see [2] and [3].
The interpretation of Grzegorczyk, see [4] and [5].
The models of Kripke, see [6] and [7].
In "Thirty years of foundational studies" A. Mostowski ([8]) gives a review of the interpretations, proposed for intuitionistic logic, on pp. 90-98.

In this paper we consider another truth-definition for formulas of intuitionistic predicate calculus. The main difference between the models treated here and Beth- and Kripke-models is, that in the new models validity is defined by validity in the elements of a spread, while in Beth- and Kripke-models validity is defined by validity in the nodes of some partially ordered set. It turns out that defining validity, as proposed in this paper, has several advantages.

Working with the new notion of validity (in completeness theorems and in modeltheory), we will use intuitionistic metamathematics, contrary to the custom to treat semantics for intuitionistic logic in a classical way.

Let $A = A(P_1, \ldots, P_k)$ be a sentence, built from the predicate-symbols $P_1, \ldots, P_k$ and from individual-symbols in the usual way.
By the structural validity of A we mean that for each domain D (for example D = IN or D is some spread) and for all predicates $P_1^*, \ldots, P_k^*$ over D, the concrete sentence $A^* = A(P_1^*, \ldots, P_k^*)$, which results from A by interpreting

the individual-symbols in A as elements (individuals) of D and by interpreting the predicate symbols $P_1, \ldots, P_k$ as the predicates $P_1^*, \ldots, P_k^*$ over D, is true.

From the definition of derivability in IPC it is clear that IPC is sound with respect to this notion of structural validity:

(i) If A is derivable in IPC, then A is structurally valid.

The notion of structural validity contains a quantification over all domains D and a quantification over all predicates $P_1^*, \ldots, P_k^*$ over D.
Although these quantifications are not quantifications over IN or over a spread, we can give a clear interpretation of the meaning of the soundness-statement (i), because the quantifications occur in the succedent of the implication. Because in the converse statement "if A is structurally valid, then A is derivable" the quantifications in the notion of structural validity occur in the antecedent of the implication, it is much more problematic to understand the meaning of this statement.

However, in this paper we will develop a notion of a model, in which all individual symbols are interpreted as individuals (elements) of IN.
A model M will interprete an m-ary predicate-symbol P as a predicate P^* over the natural numbers and at the same time will give for each m-tuple $\langle n_1, \ldots, n_m \rangle$ of natural numbers a possible development of our knowledge about the proposition $P^*(n_1, \ldots, n_m)$. It turns out that these models form a fan W and "$\models A$" will mean that A holds in all models of W.

So, in the notion of " $\models A$", we have a quantification over a fan and hence we have a precisely defined intuitionistic construction as interpretation of the completeness-statement.

(ii) if $\models A$, then A is derivable in IPC.

In the modeltheory, presented here, I also apply an idea of W. Veldman, see [10] , which enabled him to give an intuitionistically correct proof of the completeness theorem for Kripke-models.

The idea is not to restrict ourselves to the consideration of non-exploding models, but also to consider exploding models, where a Kripke-model is called exploding, if the false formula "f" does hold in it somewhere. This idea seems to give an interpretation of negation, which is nearer to what Brouwer really meant by negation: I may discover at some moment that my presuppositions yield absurdity, and hence everything.

It should be noted, that already E.W. Beth himself has felt in one way or another the necessity to consider a larger class of models than one usually does. In his book "The foundations of mathematics", page 461, [14], he speaks rather vaguely about semi-models and remarks that the completeness theorem can be intuitionistically proved, if one considers the larger set of all semi-models instead of the class of all models.

It turns out that the notion of validity, which will be introduced in chapter I, section 1 and 2, offers several advantages, compared with the other notions of validity. These advantages will be indicated at the appropriate places, namely in

Remark A, before example 1.5, chapter I
Remark B, after lemma 2.4, chapter I
Remark C, after theorem 3.2, chapter I

In chapter I we define in section 1 our notion of validity for the formulas of the intuitionistic propositional calculus and give the motivation for this definition. In section 2 we extend this definition to the predicate calculus. In section 3 we sketch two different proofs of the completeness theorem. Also two versions of the compactness theorem are proved and, using Brouwer's Continuity Principle, the formal equivalence of a Beth-model and of a model, in the sense of this paper is shown.
In section 4 we discuss the relation between our completeness result, Markov's principle and Church's Thesis .

In chapter II a detailed proof of the completeness theorem (in intuitionistic metamathematics) is given, using ideas of W. Veldman [10] and M. Fitting [18].

In chapter III "First Steps in Intuitionistic Modeltheory" we will do some modeltheory with respect to the models, defined in chapter I, and, as in the other chapters, all our metamathematical results will be proved in intuitionistic metamathematics. In section 1 we will give intuitionistically correct proofs of some theorems, which were already proved in classical metamathematics (see C.A. Smorynski [17]), e.g. the disjunction property and the explicit definability theorem. Let W be the fan of all models and let Γ be a countably infinite sequence of sentences. Then we will see in section 2 that

(i) $(A)_{A\in\Gamma}\ (EM)_{M\in W}$ (not $M \models A$) iff

$$(EM)_{M\in W}\ (A)_{A\in\Gamma} \text{ (not } M \models A)$$

(ii) $(M)_{M\in W}\ (EA)_{A\in\Gamma}\ (M \vdash A)$ iff

$$(EA)_{A\in\Gamma}(M)_{M\in W}\ (M \vdash A)$$

In chapter I, section 3, two compactness theorems are proved. In section 2 of chapter III we prove some other compactness results. The independence of the intuitionistic connectives is studied in section 3 of chapter III.

Kripke has shown (see [6], page 100) that his notion of validity can readily be formulated in terms of Kreisel's theory FC of absolutely free choice sequences. In chapter IV we will formulate our notion of validity in terms of intuitionistic analysis, not using, as Kripke does, axioms about absolutely free choice sequences.

I AN INTUITIONISTICALLY PLAUSIBLE INTERPRETATION OF INTUITIONISTIC LOGIC

§1. Models for the intuitionistic propositional calculus; Motivation.

For simplicity we start with the intuitionistic propositional calculus.

Suppose our language consists of the following symbols:

$P_1, Q_1, P_2, Q_2, \ldots.$ and f (falsity): propositional symbols

$\vee, \&, \rightarrow$ connectives.

If A is a formula, we define $- A \underset{D}{\equiv} A \rightarrow f$.

$\sigma_{01} \underset{D}{\equiv} \{0,1\}^{\mathbb{N}}$, more precisely: σ_{01} is the binary fan of all infinite sequences of zero's and one's.

$\sigma_{01\ \mathrm{mon}} \underset{D}{\equiv} \{\alpha \in \sigma_{01};\ (n)(\alpha(n+1) \geq \alpha(n))\}$.

Let PS denote the set of all propositional symbols.

Now, we first state the definitions. Afterwards we will give an elaborate motivation.

<u>Definition 1.1:</u> A <u>model</u> M is a dressed spread $<S_M, T_M>$, where S_M, the spreadlaw, regulates the choices of natural numbers, while T_M, the complementary law, assigns to each finite

sequence, which is admissible according to the spread-law S_M, a mapping from PS to $\{0,1\}$, such that for every $Q \in PS$ if $T_M(\langle a_0, \ldots, a_n \rangle)(Q) = 1$, then $T_M(\langle a_0, \ldots, a_n, a_{n+1} \rangle)(Q) = 1$.

If α is an infinitely proceeding sequence admitted by S_M then M_α: $PS \to \sigma_{01\ mon}$ is defined by $M_\alpha(Q)(k) \underset{D}{=} T_M(\bar{\alpha}k)(Q)$.

If no confusion is possible, we will sometimes use "admissible" instead of "admitted by S_M".

<u>Remark:</u> It will turn out that we can restrict ourselves to models $M = \langle S, T_M \rangle$, where S is one fixed spread-law for all M, for example S is the universal spread-law.

<u>Definition 1.2</u>: Let M be a model $\langle S_M, T_M \rangle$ and let α be an infinitely proceeding sequence, admitted by S_M.
$M_\alpha \models f \underset{D}{=}$ for some $k \in \mathbb{N}$, $M_\alpha(f)(k) = 1$.

$$M_\alpha \models A \underset{D}{=}$$

1. $M_\alpha \models f$ or
2. $A = P$ and for some $k \in \mathbb{N}$, $M_\alpha(P)(k) = 1$ or
3. $A = B \vee C$ and ($M_\alpha \models B$ or $M_\alpha \models C$) or
4. $A = B \,\&\, C$ and ($M_\alpha \models B$ and $M_\alpha \models C$) or
5. $A = B \to C$ and there is some $k \in \mathbb{N}$, such that for all admissible infinitely proceeding sequences β, if $\bar{\beta}k = \bar{\alpha}k$ and $M_\beta \models B$, then $M_\beta \models C$.

Notice, that if $M_\alpha \models f$, then $M_\alpha \models A$ for all formulas A.

Definition 1.3: Let M be a model $\langle S_M , T_M \rangle$.

$M \models A \underset{D}{\equiv} M_\alpha \models A$ for all infinitely proceeding sequences α, admitted by S_M .

Notice, that $M \models B \to C$ iff for all admissible α, if $M_\alpha \models B$, then $M_\alpha \models C$.

Lemma 1.4: Let M be a model and α an infinitely proceeding sequence, admitted by S_M.

$M_\alpha \models A$ iff there is some $k \in \mathbb{N}$, such that for all admissible infinitely proceeding sequences β, if $\bar{\beta}k = \bar{\alpha}k$, then $M_\beta \models A$.

Proof: by induction on the complexity of A.

Although in lemma 1.4 the validity in admissible infinitely proceeding sequences is reduced to validity in finite sequences of natural numbers, it turns out that the notion of validity, as defined in this paper (definition 2.8), offers several advantages, compared with the (generalized) notions of validity of Beth and Kripke, as will be indicated in the remarks A, B and C in this chapter, before example 1.5, after lemma 2.4 and after theorem 3.2, respectively.

Motivation of definition 1.1 and 1.2:

Intuitively, a model M assigns to each propositional variable P a basic-sentence $M(P)$ over the natural numbers and hence to each formula A a sentence $M(A)$ over the natural numbers, namely $M(B \,\&\, C) = M(B) \,\&\, M(C)$

$$M(B \vee C) = M(B) \vee M(C)$$
$$M(B \to C) = M(B) \to M(C)$$

and gives at the same time a possible development of our knowledge about those sentences.
More precisely, there is a spread of possibilities to search for a proof of $M(A)$, each search-path α in the spread representing a set of presuppositions for $M(A)$.

Searching along one possible path α we write

$M_\alpha(A)(i) = 1$ if I have found a proof of $M(A)$ at the time i, searching along the path α.

and $M_\alpha(A)(i) = 0$ if I have not found a proof of $M(A)$ at the time i, searching along the path α.

Now, let $M_\alpha \models A$ denote that searching along the path α I find a proof of $M(A)$, i.e. $M_\alpha(A)(i) = 1$ for some i.

Given a model M and a path α, along which we search for a proof of $M(A)$, the following conditions should be satisfied:

$M_\alpha \models B \& C$ iff searching along α I find a proof of $M(B \& C)$
iff searching along α I find a proof of $M(B)$ and searching along α I find a proof of $M(C)$
iff $M_\alpha \models B$ and $M_\alpha \models C$.

$M_\alpha \models B \vee C$ iff searching along α I find a proof of $M(B \vee C)$
iff searching along α I find a proof of $M(B)$ or searching along α I find a proof of $M(C)$
iff $M_\alpha \models B$ or $M_\alpha \models C$.

$M_\alpha \models B \rightarrow C$ iff searching along α I find a proof of $M(B) \rightarrow M(C)$
iff searching along α there is some i such that I have a proof of $M(B) \rightarrow M(C)$ at time i.

To have a proof of $M(B) \to M(C)$ at time i along the search-path α means that as soon as I have a proof of $M(B)$ along the search-path α, I also have a proof of $M(C)$ along α. Moreover: if the search-path β equals the search path α up till i, this means that also along the search-path β I have a proof of $M(B) \to M(C)$ at time i, and hence, for each such β, if I have a proof of $M(B)$ along β, then I also have a proof of $M(C)$ along β.

Hence: $M_\alpha \models B \to C$ iff there is some i such that for all β, if $\bar{\beta}i = \bar{\alpha}i$ and $M_\beta \models B$, then $M_\beta \models C$.

$M \models A$ means that $M(A)$ is true, i.e. along each search-path α I find a proof of $M(A)$ i.e. $M_\alpha \models A$ for all α, admitted by S_M.

The idea to allow in Kripke models that $M \models_s f$ (s is a finite sequence of natural numbers), under the condition "if $M \models_s f$, then $M \models_s A$ for all formulas A", is due to W. Veldman [10] .
Kripke models of this type are called generalized Kripke models. By this idea W. Veldman was able to give an intuitionistically correct completeness proof (with respect to generalized Kripke models), escaping the Gödel-Kreisel argument of [19] (see section 4).
W. Veldman's idea was applied by the author to Beth models in [11] , in which an intuitionistically correct completeness proof is given with respect to generalized Beth models.
We have applied W. Veldman's idea again in the models, defined in this paper.

"If $M_\alpha \models f$, then $M_\alpha \models A$ for all formulas A" says: if along the search-path α I discover that my presuppositions yield absurdity, then each sentence holds along the search path α. As remarked in the introduction, W. Veldman's idea seems to give an interpretation of negation, which is nearer to what Brouwer meant by negation.

Notice the difference between Intuitionistic and Minimal logic. For intuitionistic logic the condition "if $M_\alpha \models f$, then $M_\alpha \models A$ for all formulas A" should hold, but not for minimal logic.

<u>Difference between our models and Kripke- and Beth-models:</u>

Motivation for Kripke and Beth models	Motivation for our models
A model M assigns to each formula A one concrete sentence M(A) over IN and there is a partially ordered set of possible proof-situations s for M(A).	A model M assigns to each formula A one concrete sentence M(A) over IN and there is a spread of possibilities α (presuppositions) to search for a proof of M(A).
Kripke: $M \models_s A$ means that in the proofsituation s I have a proof of M(A). Beth: $M \models_s A$ means that in the proof-situation s it holds that,	$M_\alpha \models A$ means that searching along the path α I find a proof of M(A), i.e. searching along α there is some i such that at time i

no matter how I proceed with my research, I will find a proof of M(A).

I have found a proof of M(A).

Kripke: $M \models A$ means that in all proof-situations s I have a proof of M(A).

Beth: $M \models A$ means that for all s $M \models_s A$.

$M \models A$ means that along each search-path α I will find a proof of M(A).

Remark A: Because of the motivation for Beth-models and for our models one can expect that a generalized Beth-model is formally equivalent with a model in our sense. Using Brouwer's Continuity Principle this turns out to be true (theorem 3.5).

One might say that in the models, as defined in this paper, the motivation behind the Beth models appears more precisely and more explicitly than in the Beth models themselves.

Notice the difference between $M \models A$ for Kripke-models at the one hand and $M \models A$ for Beth- and for our models at the other hand. $M \models A$ in the sense of Kripke means that in all proofsituations s I have a proof of M(A), while $M \models A$ in the sense of Beth or in our sense rather means that along each search-path α I will find a proof of M(A).

Remark: In definition 1.1 we defined a dressed spread $\langle S_M, T_M \rangle$ by adjoining to each finite sequence, admitted by S_M, an element of another spread, namely an element of $\{0,1\}^{PS}$.

However, it is not necessary to use in the definition of a model another spread, because in practice we are interested in the models for some formula A, in which only finitely many propositional variables occur.

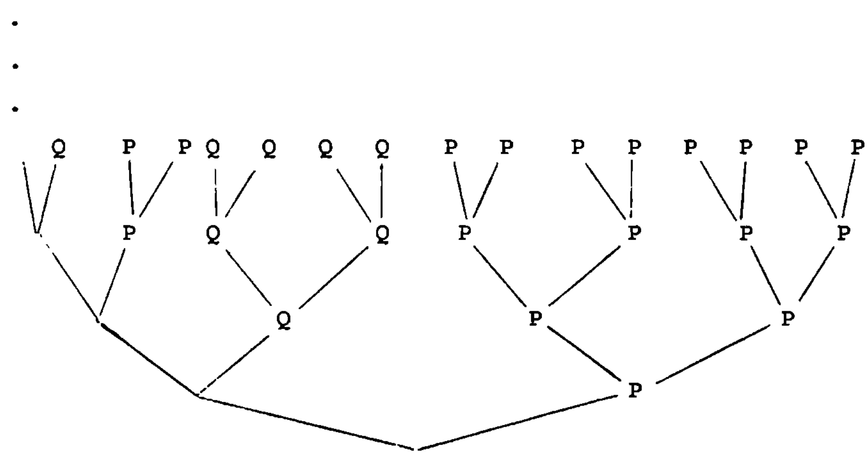
So we can replace definition 1.1 by the following one:

A model M is a dressed spread $<S_M , T_M>$, where S_M, the spread-law, regulates the choices of natural numbers, while T_M, the complementary law, assigns to each finite sequence, which is admissible according to the spread-law S_M, a finite subset of PS, such that if $Q \in T_M(<a_0 ,\ldots a_n>)$, then $Q \in T_M(<a_0,\ldots,a_n ,a_{n+1}>)$.

If α is an admissible infinitely proceeding sequence, then M_α: PS $\rightarrow \sigma_{01\ mon}$ is defined by $M_\alpha(Q)(k) = 1 \underset{D}{\equiv} Q \in T_M(\bar{\alpha}k)$

$= 0$ otherwise.

<u>Example 1.5</u> : Let $M = <S_M ,T_M>$ be the following model:

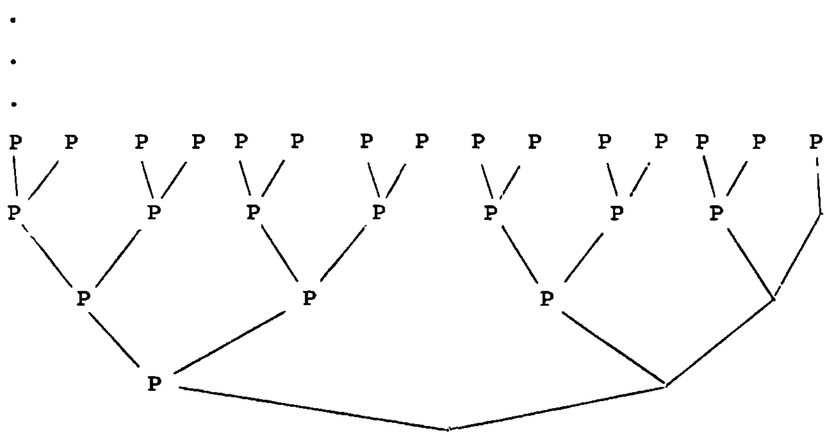


Then: not $M \models -(P \,\&\, Q) \rightarrow -P \vee -Q$.

Proof: Suppose $M \models -(P \,\&\, Q) \rightarrow -P \vee -Q$.
Then for all α, if $M_\alpha \models -(P \,\&\, Q)$, then $M_\alpha \models -P \vee -Q$.
Now for all α, $M_\alpha \models -(P \,\&\, Q)$. Hence for all α, $M_\alpha \models -P$ or $M_\alpha \models -Q$.
But for β the left-most admissible infinitely proceeding sequence, not $M_\beta \models -P$ and not $M_\beta \models -Q$. Contradiction.

q.e.d.

Example 1.6: Let $M = \langle S_M, T_M\rangle$ be the following model.



Then: not $M \models --P \rightarrow P$.

Proof: Suppose $M \models --P \rightarrow P$.
Then for all α, if $M_\alpha \models --P$, then $M_\alpha \models P$.
Now, for all α, $M_\alpha \models --P$. Hence, for all α, $M_\alpha \models P$.
But for β the right-most admissible infinitely proceeding sequence, not $M_\beta \models P$.
Contradiction. q.e.d.

§2. Models for the intuitionistic predicate calculus.

Let τ: IN $\rightarrow$ IN be given, with $\tau(o) = 0$.

Suppose the alphabet of our language has the following symbols:

$v_1, v_2, \ldots$	free variables
$x_1, x_2, \ldots$	bound variables
$P_1, P_2, \ldots$ }	predicate symbols
$f = P_o$ (falsity)	
$\vee, \&, \rightarrow$	connectives
(x), (Ex)	quantifiers

We will use v for arbitrary free variables v_i with $i = 1,2,..$ x for arbitrary bound variables x_i with $i = 1,2 \ldots$ and P and Q for arbitrary predicate symbols P_i with $i \in$ IN.

P_i is an m - ary predicate symbol $\underset{D}{\equiv}$ $\tau(i) = m$.
By definition f is o-ary.

If A is a formula, we define $-A \underset{D}{\equiv} A \rightarrow f$.

σ_{01} is the binary fan of all infinite sequences of zero's and one's.

$\sigma_{01\,\mathrm{mon}} \underset{D}{\equiv} \{\alpha \in \sigma_{01};\ (n)\ (\alpha(n+1) \geqslant \alpha(n))\}$.

PN $\underset{D}{\equiv}$ the set of all expressions of the form
f or $< Q, n_1, \ldots, n_m >$
where m is an arbitrary natural number, Q is an m-ary predicate symbol and $n_1, \ldots n_m$ are natural numbers.

Notation: $A(v_1, \ldots, v_m)$ is a formula A, such that each free variable, occurring in A, is an element of $\{v_1, \ldots, v_m\}$.

A model M interpretes an m-ary predicate symbol P as a predicate P^* over the natural numbers and gives at the same time for each m-tuple $\langle n_1, \ldots, n_m \rangle$ of natural numbers a possible development of our knowledge about the proposition $P^*(n_1, \ldots, n_m)$.

<u>Definition 2.1</u>: A <u>model</u> M is a dressed spread $\langle S_M, T_M \rangle$, where S_M, the spread-law, regulates the choices of natural numbers, while T_M, the complementary law, assigns to each finite sequence, which is admissible according to the spread-law S_M, a mapping from PN to $\{0,1\}$, such that, if $T_M(\langle a_o, \ldots, a_k \rangle)(\langle Q, n_1, \ldots, n_m \rangle) = 1$, then $T_M(\langle a_o, \ldots, a_k, a_{k+1} \rangle)(\langle Q, n_1, \ldots, n_m \rangle) = 1$.

If α is an infinitely proceeding sequence, admitted by S_M, then M_α: PN $\rightarrow \sigma_{01\ mon}$ is defined by $M_\alpha(\langle Q, n_1, \ldots, n_m \rangle)(k) \underset{D}{=} T_M(\bar{\alpha}k)(\langle Q, n_1, \ldots, n_m \rangle)$.

<u>Remark:</u> It will turn out that we can restrict ourselves to models $\langle S, T_M \rangle$, where S is one fixed spread-law for all M, for example the universal or the binary spread-law.

<u>Definition 2.2</u>: Let M be a model $\langle S_M, T_M \rangle$ and let α be an infinitely proceeding sequence, admitted by S_M.

$M_\alpha \models f \underset{D}{=}$ for some $k \in$ IN, $M_\alpha(f)(k) = 1$.

Let $A(v_1, \ldots, v_m)$ be a formula and let $n_1, \ldots, n_m$ be natural numbers.

$M_\alpha \models A(v_1, \ldots, v_m)[n_1, \ldots, n_m] \underset{D}{=}$

1. $M_\alpha \models f$ or
2. $A(v_1, \ldots, v_m) = P(v_1, \ldots, v_m)$, m an arbitrary natural number, P an m-ary predicate symbol, and for some $k \in$ IN

$M_\alpha(\langle P, n_1, \ldots, n_m \rangle)(k) = 1$, or

3. $A = B \vee C$ and ($M_\alpha \models B[n_1, \ldots, n_m]$ or $M_\alpha \models C[n_1, \ldots, n_m]$) or
4. $A = B \,\&\, C$ and ($M_\alpha \models B[n_1, \ldots, n_m]$ and $M_\alpha \models C[n_1, \ldots, n_m]$) or
5. $A = B \rightarrow C$ and there is some $k \in \mathbb{N}$, such that for all admissible infinitely proceeding sequences β with $\bar{\beta}k = \bar{\alpha}k$, if $M_\beta \models B[n_1, \ldots, n_m]$, then $M_\beta \models C[n_1, \ldots, n_m]$, or
6. $A(v_1, \ldots, v_m) = (Ex)B(x, v_1, \ldots, v_m)$ and there is some $n \in \mathbb{N}$ such that $M_\alpha \models B(v, v_1, \ldots, v_m)[n, n_1, \ldots, n_m]$, or
7. $A(v_1, \ldots, v_m) = (x)B(x, v_1, \ldots, v_m)$ and there is some $k \in \mathbb{N}$, such that for all admissible infinitely proceeding sequences β with $\bar{\beta}k = \bar{\alpha}k$ and for all natural numbers n $M_\beta \models B(v, v_1, \ldots, v_m)[n, n_1, \ldots, n_m]$.

<u>Definition 2.3</u>: Let M be a model $\langle S_M, T_M \rangle$.
$M \models A(v_1, \ldots, v_m)[n_1, \ldots, n_m] \underset{D}{\equiv}$ for all infinitely proceeding sequences α,
admitted by S_M, $M_\alpha \models A(v_1, \ldots, v_m)[n_1, \ldots, n_m]$.

<u>Lemma 2.4</u>: Let M be a model and α an infinitely proceeding sequence, admitted by S_M.
$M_\alpha \models A(v_1, \ldots, v_m)[n_1, \ldots, n_m]$ iff there is some $k \in \mathbb{N}$, such that for all infinitely proceeding sequences β, admitted by S_M, if $\bar{\beta}k = \bar{\alpha}k$, then $M_\beta \models A(v_1, \ldots, v_m)$ $[n_1, \ldots, n_m]$.

Proof: by induction on the complexity of A.

Remark B: Notice, that we have defined validity by validity in admissible infinitely proceeding sequences α ($M_\alpha \models A$), while in Beth-, Grzegorczyk- and Kripke- models validity is defined by validity in the nodes s of some partially ordered set ($M \models_s A$).

This causes, that from a technical point of view our definition of $M \models A(v_1, \dots, v_m)[n_1, \dots, n_m]$ is simpler than the corresponding notions of Beth [2] and Kripke [6] :

For Beth-models the satisfaction-relation is complicated in the case of atomic formulas, disjunction, implication and existential quantifier, but they have a fixed universe. In Kripke-models the universe is growing and the satisfaction-relation is complicated in the case of implication and universal quantifier.

Now, in our new semantics, the universe is a fixed one, namely IN, and all cases in the definition of the satisfaction-relation are straightforward, except in the case of implication and universal quantifier.

It should be noted here that each model M in our sense can be conceived as a topological model in the sense of [1] :

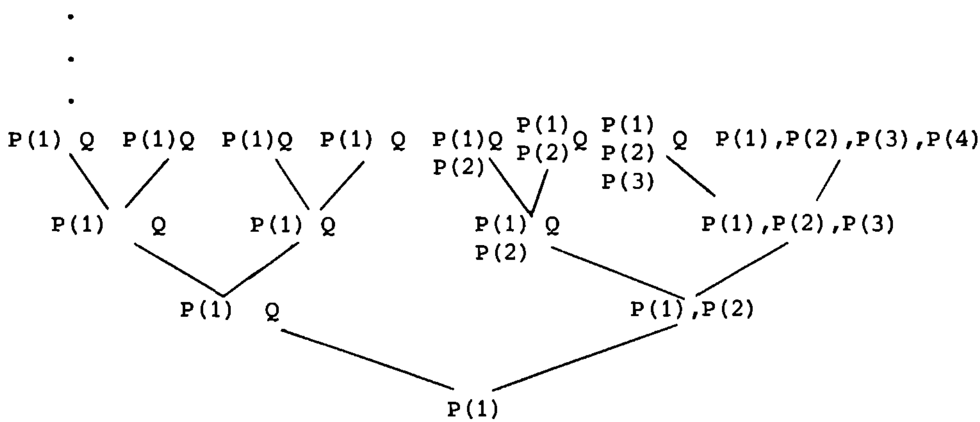
Take for the underlying topological space the admissible infinitely proceeding sequences as points and the sets $V_s = \{\alpha \mid \bar{\alpha} k = s \text{ for some } k \in \mathbb{N}\}$ as neighbourhoods (s is an admitted finite sequence of natural numbers).

The open set [[A]], associated with the sentence A in the topological model is $\{\alpha \mid M_\alpha \models A\}$.

$$\begin{aligned}\text{Then } [[B \vee C]] &= \{\alpha \mid M_\alpha \models B \vee C\} \\ &= \{\alpha \mid M_\alpha \models B \text{ or } M_\alpha \models C\} \\ &= \{\alpha \mid M_\alpha \models B\} \cup \{\alpha \mid M_\alpha \models C\} \\ &= [[\,B\,]] \cup [[\,C\,]]\end{aligned}$$

So, the technical simplification, with respect to Beth-models, mentioned above in for example the interpretation of $\vee$, was already known in classical metamathematics (see [1]), because of the well-known equivalence of Beth-models and topological models (see [3] and [21] , section 4).

<u>Example 2.5</u>: Let $M = \langle S_M, T_M \rangle$ be the following model:



Then: not $M \models (x)(P(x) \vee Q) \rightarrow (x)(P(x)) \vee Q$.

Remark: Let β be the rightmost admissible, infinitely proceeding sequence.

Then $M_\beta(Q)(k) = 0$ for all $k \in \mathbb{N}$.

$M_\beta(P(1)) = 1\ 1\ 1\ 1\ \ldots$

$M_\beta(P(2)) = 0\ 1\ 1\ 1\ \ldots$

$M_\beta(P(3)) = 0\ 0\ 1\ 1\ \ldots$

$M_\beta(P(4)) = 0\ 0\ 0\ 1\ \ldots$

Proof: Suppose $M \models (x)(P(x) \vee Q) \rightarrow (x)(P(x)) \vee Q$.
Then for all α, if $M_\alpha \models (x)(P(x) \vee Q)$, then $M_\alpha \models (x)(P(x)) \vee Q$.
$M_\beta \models (x)(P(x) \vee Q)$. Hence $M_\beta \models (x)(P(x))$ or $M_\beta \models Q$.
But not $M_\beta \models (x)(P(x))$ and not $M_\beta \models Q$.
Contradiction. q.e.d.

<u>Definition 2.6</u>: $\mathrm{IN}^\infty \underset{D}{\equiv} \{ a \in \mathrm{IN}^{\mathrm{IN}};\ (En)(k)(k > n \rightarrow a_k = a_n) \}$.

Let $M = \langle S_M, T_M \rangle$ be a model, $A = A(v_{i_1}, \ldots, v_{i_m})$ a formula, with m free variables, and $a \in \mathrm{IN}^\infty$.

$M_\alpha \models A[a] \underset{D}{\equiv} M_\alpha \models A(v_{i_1}, \ldots, v_{i_m})[a_{i_1}, \ldots a_{i_m}]$.

$M_\alpha \models A \underset{D}{\equiv} M_\alpha \models A[a]$ for all $a \in \mathrm{IN}^\infty$.

$M \models A \underset{D}{\equiv} M_\alpha \models A$ for all infinitely proceeding sequences α, admitted by S_M.

<u>Theorem 2.7</u>: The models $M = \langle S, T_M \rangle$, where S is one fixed spread-law for all M, form a fan W.

<u>Proof</u>: Let $d_0, d_1, d_2, \ldots$ be an enumeration of PN and let $s_0, s_1, s_2, \ldots$ be an enumeration of all admissible finite sequences, such that if s_i is an initial segment of s_j, then $i \leq j$.

We can represent a model $M = \langle S, T_M \rangle$ by the following scheme:

	d_0	d_1	d_2	d_3	...
$T_M(s_0)$	0	1	0	1	...
$T_M(s_1)$	.	.	.	.	
$T_M(s_2)$	.	.	.	.	
.					
.					
.					

where the following local conditions should be satisfied:
is s_i is an initial segment of s_j, then $T_M(s_i)(k) \leqslant T_M(s_j)(k)$ for all $k \in \mathrm{IN}$.
So we can code a model $M = \langle S, T_M \rangle$ as an infinite sequence μ of natural numbers with $\mu(\langle s_i, d_j \rangle) = T_M(s_i)(d_j)$.
Now it is easy to see that the theorem holds.

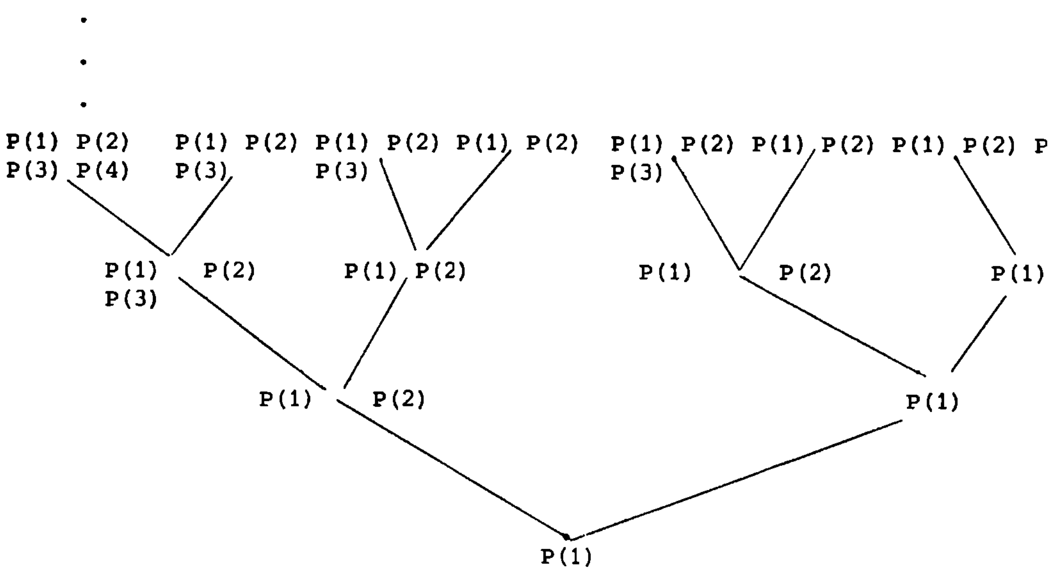
q.e.d.

<u>Definition 2.8</u>: Let A be a formula and Γ a (possibly infinite) set of formulas.

$\models A \underset{D}{\equiv} M \models A$ for all $M \in W$.

$\Gamma \models A \underset{D}{\equiv}$ for all $M \in W$, for all $a \in \mathrm{IN}^{\infty}$ and for all admissible infinitely proceeding sequences α, if $M_\alpha \models C\,[a]$ for all $C \in \Gamma$, then $M_\alpha \models A\,[a]$.

<u>Example 2.9</u>: Let $M = \langle S_M, T_M \rangle$ be the following model:



Then: $M \models -(x)(P(x) \vee -P(x))$.

Proof: It is sufficient to show, that for each α, not $M_\alpha \models (x)(P(x) \lor -P(x))$.

So, suppose α is an admissible infinitely proceeding sequence and $M_\alpha \models (x)(P(x) \lor - P(x))$. Then there is some $k \in \mathbb{N}$, such that for all β and for all $n \in \mathbb{N}$, if $\bar{\beta}k = \bar{\alpha}k$, then $M_\beta \models P(v)[n]$ or $M_\beta \models - P(v)[n]$.

Now, let γ be the rightmost admissible infinitely proceeding sequence through $\bar{\alpha}k$.

Then for all $n > k+1$, not $M_\gamma \models P(v)[n]$.

Hence $M_\gamma \models - P(v)[n]$ for all $n > k+1$.

So for all $n > k+1$ there is some $l_n \in \mathbb{N}$, such that for all δ, if $\bar{\delta}l_n = \bar{\gamma}l_n$ then not $M_\delta \models P(v)[n]$.

But for the left-most δ through $\bar{\gamma}l_{k+2}$ we do have that $M_\delta \models P(v)[k+2]$.

Contradiction. q.e.d.

Remark: The same model makes clear, that $(x)(- -P(x)) \rightarrow - - (x)(P(x))$ is not valid.

Definition 2.10: Let $M = \langle S_M, T_M \rangle$ be a model.

> M explodes $\equiv_D$ for some infinitely proceeding sequence α, admitted by S_M, $M_\alpha \models f$.
>
> M is trivial $\equiv_D$ $M \models f$, i.e. $M_\alpha \models f$ for all α.

Definition 2.11: Let Γ be a set of formulas.

> Γ is decidable $\equiv_D$ we have a method, such that for each formula A we can decide in finitely many steps whether $A \in \Gamma$ or not $A \in \Gamma$.

Later on, for the proof of the completeness of IPC with respect to the notion of validity of definition 2.8, we need the following

<u>Definition 2.12</u>: Let M be a model $<S_M, T_M>$ and let α be an admissible infinitely proceeding sequence. Let A be a formula and $a \in \mathbb{N}^\infty$.

$M_\alpha \models' A[a] \underset{D}{\equiv}$

1. M explodes or
2. $A = P(v_1, \ldots, v_m)$, m an arbitrary natural number, P an m-ary predicate symbol, and for some $k \in \mathbb{N}$, $M_\alpha(<P, a_1, \ldots, a_m>)(k) = 1$, or
3. $A = B \vee C$ and $(M_\alpha \models' B[a]$ or $M_\alpha \models' C[a])$, or
4. $A = B \& C$

.

.

.

7. $A = (x)B(x, v_1, \ldots, v_m)$

(4. – 7.) as in definition 2.2 with " $\models'$ " instead of " $\models$ ".

$\models' A \underset{D}{\equiv}$ for all $M \in W$, for all α and for all $a \in \mathbb{N}^\infty$, $M_\alpha \models' A[a]$.

Let Γ be a set of formulas.

$\Gamma \models' A \underset{D}{\equiv}$ for all $M \in W$, for all α and for all $a \in \mathbb{N}^\infty$, if $M_\alpha \models' C[a]$ for all $C \in \Gamma$, then $M_\alpha \models' A[a]$.

<u>Lemma 2.13</u>: $M_\alpha \models' A[a]$ iff there is some $k \in \mathbb{N}$, such that for all β, if $\bar{\beta}k = \bar{\alpha}k$, then $M_\beta \models' A[a]$.

Proof: By induction on the complexity of A.

q.e.d.

§3. Soundness, completeness and compactness.

Theorem 3.1 (Soundness): Let A be a formula and Γ a set of formulas.

i) If $\vdash_{IPC} A$, then $\models A$.
ii) If $\Gamma \vdash_{IPC} A$, then $\Gamma \models A$.

Proof: by induction on the length of a derivation of A .

We will sketch now two intuitionistic completeness-proofs. The first one is an adaptation of W. Veldman's completeness proof for generalized Kripke-models in [10] . The second one is an adaptation of the author's completeness proof for generalized Beth-models in [11] . After a sketch in this chapter, this second proof will be given in full detail in chapter II.

Theorem 3.2 (Completeness): Let A be a formula and Γ a decidable (possibly infinite) set of formulas (see definition 2.11)

i) There is a model $M = \langle S_M, T_M \rangle$, where S_M determines a spread, which is not a fan, such that if $M \models A(v_{i_1}, \dots, v_{i_m})[i_1, \dots, i_m]$, then $\vdash_{IPC} A(v_{i_1}, \dots, v_{i_m})$.
Hence, if $\models A$, then $\vdash_{IPC} A$.
ii) There is a model $M = \langle S_M, T_M \rangle$, where S_M determines a spread, which is not a fan, such that $M \models C(v_{i_1}, \dots, v_{i_m})[i_1, \dots, i_m]$ for all

$C(v_{i_1},\ldots,v_{i_m}) \in \Gamma$ and such that if $M \models A(v_{i_1},\ldots,v_{i_m})$ $[i_1,\ldots,i_m]$, then $\Gamma \vdash_{IPC} A$.

Hence, if $\Gamma \models A$, then $\Gamma \vdash_{IPC} A$.

Remark C: In [10] W. Veldman constructed a spread such that to each α there was associated a semi-regular set Γ_α of sentences.

So, what he actually constructed, was precisely a model in our sense.

In order to make a Kripke-model from this, he had to take the elements of the spread as the nodes of a set, which is partially ordered by $\alpha \leqslant \beta \underset{D}{\equiv} \Gamma_\alpha \subseteq \Gamma_\beta$. Because this ordering is non-discrete, this was a rather strange Kripke-model.

It is nice to see, that with our notion of a model, the proof of [10], after adaptation, becomes more natural: our models are dressed spreads and not trees with a partial ordering.

Instead of $\Gamma \vdash_{IPC} A$ we will sometimes simply write: $\Gamma \vdash A$.

Sketch of a proof: i) Let $M = \langle S_M, T_M \rangle$ be the following model: S_M is the spread-law Σ, defined in [10], 3.32 and $T_M(\bar{\alpha}k)(\langle P, n_1,\ldots,n_m\rangle) = 1 \underset{D}{\equiv} P(v_{n_1},\ldots,v_{n_m}) \in \Gamma(\bar{\alpha}k)$, where Γ is defined as in [10], 3.32.

As in [10], it is now sufficient to show that for all α, admitted by S_M and for all formulas $A(v_{i_1},\ldots,v_{i_m})$

$$M_\alpha \models A(v_{i_1},\ldots,v_{i_m})[i_1,\ldots,i_m] \text{ iff } A(v_{i_1},\ldots,v_{i_m}) \in \Gamma_\alpha .$$

$A = P(v_{i_1}, \ldots, v_{i_m}) : M_\alpha \models P(v_{i_1}, \ldots, v_{i_m})[i_1, \ldots, i_m]$

$\Leftrightarrow$ for some k, $T_M(\bar{\alpha}k)(\langle P, i_1, \ldots, i_m\rangle) = 1$

$\Leftrightarrow$ for some k, $P(v_{i_1}, \ldots, v_{i_m}) \in \Gamma(\bar{\alpha}k)$

$\Leftrightarrow$ $P(v_{i_1}, \ldots, v_{i_m}) \in \Gamma_\alpha$.

$A = B \to C$, from right to left: Suppose $B \to C \in \Gamma_\alpha$.
Then for some k, $B \to C \in \Gamma(\bar{\alpha}k)$.

We want to show that if $\bar{\beta}k = \bar{\alpha}k$ and $M_\beta \models B$, then $M_\beta \models C$.
So suppose $\bar{\beta}k = \bar{\alpha}k$ and $M_\beta \models B$. By induction hypothesis $B \in \Gamma_\beta$. Because $B \to C \in \Gamma(\bar{\alpha}k)$ and $\bar{\beta}k = \bar{\alpha}k$, $B \to C \in \Gamma_\beta$. Hence $\Gamma_\beta \vdash C$. So $C \in \Gamma_\beta$ and by induction hypothesis $M_\beta \models C$.

$A = B \to C$, from left to right: Suppose $M_\alpha \models B \to C$,
i.e. there is some k such that for all β,
if $\bar{\beta}k = \bar{\alpha}k$ and $M_\beta \models B$, then $M_\beta \models C$.

By induction hypothesis: there is some k, such that for all β, if $\bar{\beta}k = \bar{\alpha}k$ and $B \in \Gamma_\beta$, then $C \in \Gamma_\beta$.
Let F be the subfan, defined in 4.1 of [10] , restricted to those β, for which $\bar{\beta}k = \bar{\alpha}k$.
Then $C \in \Gamma_\beta$ for all $\beta \in F$, i.e. for all $\beta \in F$ there is an $m \in \mathbb{N}$ such that $C \in \Gamma(\bar{\beta}m)$.
By the fan-theorem there is some $m \in \mathbb{N}$, such that for all $\beta \in F$, $C \in \Gamma(\bar{\beta}m)$.
Like in 4.1 of [10] , we can conclude that $\Gamma_\alpha, B \vdash C$.
So $\Gamma_\alpha \vdash B \to C$ and hence $B \to C \in \Gamma_\alpha$.

The other cases are similar, see [10].

The proof of ii) is an easy generalization of the proof of i).

q.e.d.

As a consequence of theorem 3.2 we get a compactness theorem for non-trivial models (see definition 2.10):

Corollary 3.3 (Compactness): Let Γ be a decidable (possibly infinite) set of sentences.

If each finite subset of Γ has a non-trivial model, then Γ has a non-trivial model.

Proof: Suppose each finite subset of Γ has a non-trivial model. Then not $\Gamma \vdash f$.
Let $M = \langle S_M, T_M \rangle$ be the model of theorem 3.2 ii). Then for all $C \in \Gamma$, $M \models C$, and if $M \models f$, then $\Gamma \vdash f$. Hence not $M \models f$, i.e. M is non-trivial.
q.e.d.

Corollary 3.4: Let Γ be a set of formulas.
If for all models $M \in W$ (W is the fan of all models, see theorem 2.7) there is a formula $A \in \Gamma$, such that $M \models A$ ($M \models - A$), then there is some formula $A \in \Gamma$ such that $M \models A$ ($M \models - A$) for all models $M \in W$.

Proof: Suppose for all models $M \in W$ there is a formula $A \in \Gamma$, such that $M \models A$.
Then also for the model M of theorem 3.2 i) there is some formula $B \in \Gamma$ such that $M \models B$.
But then, by theorem 3.2 i) $\vdash B$, and hence, by theorem 3.1, $\models B$.
q.e.d.

In theorem 3.5 below, we will show that we can re-interprete each Beth-model as a model in the sense of definition 2.1 and 2.2. and conversely that we can re-interprete each model in the sense of these definitions as a Beth-model.

One might say that a model in the sense of our definitions is just an intuitionistic interpretation of a Beth-model and that a Beth-model is just a classical interpretation of a model in our sense.

We will show below that the notions of Beth-validity, denoted by $\models^{B}$, and of intuitionistic validity in our sense, formally coincide.

The formal equivalence of our models with Beth-models is not simply true by definition, but the proof uses Brouwer's Continuity Principle. For Beth models we take the notation of [11] and for simplicity we will restrict ourselves to non-exploding models.

Theorem 3.5 (Formal equivalence of a non-exploding Beth-model and of a non-exploding model in our sense):

i) Let $M = \langle S_M, T_M \rangle$ be a model (in our sense).
Define the Beth-model M' as follows: for s an admissible finite sequence
$M'(s) \underset{D}{=} \{\langle P, n_1, \ldots, n_m \rangle ; T_M(s)(\langle P, n_1, \ldots, n_m \rangle) = 1\}$.
Then $M' \models^{B}_{s} A[a]$ iff for all admissible α through s, $M_\alpha \models A[a]$.
Hence, if $\Gamma \models^{B} A$, then $\Gamma \models A$.

ii) Let M' be a Beth-model.
Define $M = \langle S_M, T_M \rangle$ as follows: for s an admissible finite sequence
$T_M(s)(\langle P, n_1, \ldots, n_m \rangle) = 1 \underset{D}{=} \langle P, n_1, \ldots, n_m \rangle \in M'(s)$.
Then $M' \models^{B}_{s} A[a]$ iff for all α through s , $M_\alpha \models A[a]$.
Hence, if $\Gamma \models A$, then $\Gamma \models^{B} A$.

Proof:

$A = P(v_1, \ldots, v_m)$: $M' \models^{B}_{s} P(v_1, \ldots, v_m)[a_1, \ldots, a_m]$

$\Leftrightarrow$ for all α through s there is some $k \in \mathbb{N}$ such that $\langle P, a_1, \ldots, a_m \rangle \in M'(\bar{\alpha}k)$

$\Leftrightarrow$ for all α through s there is some $k \in \mathbb{N}$ such that $T_M(\bar{\alpha}k)(\langle P, a_1, \ldots, a_m \rangle) = 1$

$\Leftrightarrow$ for all α through s, $M_\alpha \models P(v_1, \ldots, v_m)[a_1, \ldots, a_m]$.

$A = B \vee C$: $M' \models^{B}_{s} B \vee C$

$\Leftrightarrow$ for all α through s there is some $k \in \mathbb{N}$ such that $M' \models^{B}_{\bar{\alpha}k} B$ or $M' \models^{B}_{\bar{\alpha}k} C$

$\Leftrightarrow$ for all α through s there is some $k \in \mathbb{N}$ such that for all β through $\bar{\alpha}k$, $M_\beta \models B$ or for all β through $\bar{\alpha}k$, $M_\beta \models C$

(*) $\Leftrightarrow$ for all α through s, $M_\alpha \models B$ or $M_\alpha \models C$

$\Leftrightarrow$ for all α through s, $M_\alpha \models B \vee C$.

To prove (*) from below to above we use the continuity-principle of Brouwer (see [13]).

$A = B \& C$: immediate from the induction hypothesis.

$A = B \rightarrow C$: $M' \models^{B}_{s} B \rightarrow C$

$\Leftrightarrow$ for all s', such that s is an initial part of s', if $M' \models^{B}_{s'} B$, then $M' \models^{B}_{s'} C$

$\Leftrightarrow$ for all s', such that s is an initial part of s', if for all α through s', $M_\alpha \models B$, then for all α through s', $M_\alpha \models C$

(*) $\Leftrightarrow$ for all α through s, if $M_\alpha \models B$, then $M_\alpha \models C$

$\Leftrightarrow$ for all α through s, $M_\alpha \models B \rightarrow C$.

proof of (*) from above to below: suppose α goes through s and $M_\alpha \models B$; then there is some $k \geqslant$ length of s, such that for all β through $\bar{\alpha}k$, $M_\beta \models B$; hence for all β through $\bar{\alpha}k$, $M_\beta \models C$; so $M_\alpha \models C$.

$\left.\begin{array}{l} A = (Ex)B(x) \\ A = (x)B(x) \end{array}\right\}$ immediate from the induction hypothesis.

q.e.d.

Having shown the formal equivalence of (non-exploding) Beth-models and of our (non-exploding) models, we can use the results of [11], in which an intuitionistic completeness proof is given with respect to generalized Beth-models, in order to establish another completeness proof.

The advantage of this completeness proof is, that, in practice, it gives you a countermodel for those formulas, which are not valid.

But in [11] our satisfaction-relation was defined in such a way, that if M explodes (somewhere) (see definition 2.10), then all formulas A are valid in M.

So, in order to be able to adapt the completeness proof of [11], we need the following theorem:

<u>Theorem 3.6</u>: i) If $\models A$ (definition 2.8), then $\models' A$ (definition 2.12)

ii) If $\Gamma \models A$ (definition 2.8), then $\Gamma \models' A$ (definition 2.12)

Proof: Let $N = \langle S_N, T_N \rangle$ be a model.

Define $M = \langle S_M, T_M \rangle$ as follows:

$S_M = S_N$.

Let $s_1, s_2, s_3, \ldots$ be an enumeration of the admissible finite sequences.

For $P \neq f$ let $T_M(s_i)(\langle P, n_1, \ldots, n_m \rangle) \underset{D}{=} T_N(s_i)(\langle P, n_1, \ldots, n_m \rangle)$ and $T_M(s_i)(f) = 1 \underset{D}{=}$ for some $j \leq i$, $T_N(s_j)(f) = 1$.

In words: as soon as we notice f somewhere in N, then we put f in each admissible sequence in M. Now, by induction on the complexity of A one easily shows:

$$M_\alpha \models A[a] \quad \text{iff } N_\alpha \models' A[a]$$

Hence, if $\models A$, then $\models' A$, and, if $\Gamma \models A$, then $\Gamma \models' A$.

q.e.d.

In theorem 3.2 one model M is given such that validity in M implies derivability in IPC.

Hence, if $\models A$, then $\vdash_{IPC} A$.

Theorem 3.5, theorem 3.6 and the completeness proof of [11] suggest another proof of the completeness of IPC with respect to the semantics of this paper: validity in all models of a certain type implies derivability in IPC.

<u>Theorem 3.7 (Completeness)</u>: Let A be a formula and let Γ be a decidable set of formulas.

i) if $\models A$, then $\vdash_{IPC} A$.

ii) if $\Gamma \models A$, then $\Gamma \vdash_{IPC} A$.

Sketch of a proof: Suppose $\Gamma \models A$. Then, by theorem 3.6, $\Gamma \models' A$. Because of theorem 3.5 one can adapt the completeness proof of [11] to a proof of $\Gamma \vdash_{IPC} A$.

q.e.d.

The proof of theorem 3.7, which we have sketched here, will be given in full detail in chapter II.

We already mentioned that the advantage of the completeness proof of [11] is, that, in practice, it yields a counter-model for those formulas, which are not valid.
For the propositional calculus this completeness proof even gives a finite decisionprocedure: If A is a formula of the intuitionistic propositional calculus, then either A is derivable, or there is a counter-model for A.
As a consequence of theorem 3.2 we had in corollary 3.3 a compactness theorem for non-trivial models. The completeness proof of theorem 3.7 yields a weak compactness result.

<u>Corollary 3.8 (Weak Compactness)</u>: Let Γ be a decidable set of sentences. If each finite subset of Γ has a non-exploding model, then not for all $M \in W$, if M is a model of Γ, then M explodes.

Sketch of a proof: Suppose each finite subset of Γ has a non-exploding model. Then not $\Gamma \vdash f$. Now, suppose for all $M \in W$, if M is a model of Γ, then M explodes. Then all models, obtained by application of the systematic procedure, which is described in [11] and in definition 2.2 of chapter II of this paper, to $T\,\Gamma$, $F\,f$, explode.
But then $\Gamma \vdash f$. Contradiction.

q.e.d.

§4 Completeness, Markov's principle and Church's Thesis.

To allow the false formula "f" to be satisfied in a model M enlarges the class of all models and has important consequences concerning negation, as is also pointed out in W. Veldman [10] , section 5.

For a formula A, $\neg A$ is defined by $A \to f$, and hence $M \models \neg A$ means something like "if $M \models A$, then $M \models f$" , which is different from "not $M \models A$" or "if $M \models A$, then contradiction", as it is usually defined.
This causes that the Gödel-Kreisel-argument [19] , that completeness of the intuitionistic predicate calculus implies Markov's principle, cannot be applied for our notion of validity. More precisely: the proof of corollary 4.5 in D. van Dalen, "Lectures on Intuitionism" in the proceedings of the Cambridge Summer School in Mathematical Logic, 1971, [15] , does not hold for our notion of a model.
Because corollary 4.5 of [15] is used to show that the intuitionistic predicate calculus is incomplete, if Church's Thesis is assumed (page 86), also the proof of this result does not hold for our notion of validity.

All arguments fail by the essential difference in the treatment of negation.
Confer also A. Troelstra, [16], B 15.
In this context it may be worth-while to mention that in [20], page 145, G. Kreisel proves that Church's Thesis is inconsistent with the Brouwer-Kripke-scheme, attributing the result to S. Kripke.

Related results, which were pointed out by W. Gielen (Nijmegen) [9] are the following:

a) The Brouwer-Kripke-scheme

BKS $(E\alpha)_{\alpha \in \sigma_{01mon}} (A \leftrightarrow (En)(\alpha(n) = 1))$

as a scheme for with finite information completely described A, yields a function $\alpha \in \{0,1\}^{\mathbb{N}}$, which is not recursive.

Proof: BKS yields a function

$\alpha_o \in \sigma_{01mon}$ with $-(Ez)T_1(0,0,z) \leftrightarrow (En)(\alpha_o(n) = 1)$

$\alpha_1 \in \sigma_{01mon}$ with $-(Ez)T_1(1,1,z) \leftrightarrow (En)(\alpha_1(n) = 1)$

and so on.

Hence we can construct step by step a function $\alpha \in \{0,1\}^{\mathbb{N}}$ by taking $\alpha(<p,q>) = \alpha_p(q)$. The supposition that α is recursive yields a contradiction, because $-(Ez)\ T_1\ (p,p,z)$ is not recursively enumerable.

b) Church's Thesis (CT)

$(p)(Eq)B(p,q) \rightarrow (Ee)\ [\ (r)(Es)T_1(e,r,s)\ \&\ (p)B(p,\{e\}(p))]$

B without other free variables than p and q, can be refuted with BKS, if we accept the functions, which are indicated by the following notation: α_A is the choice-sequence, we get via BKS applied to A.

Proof: Apply Church's Thesis to $B(p, q) \underset{D}{\equiv}$

$$q = \alpha_{-(Ez)\, T_1((p)_o,\, (p)_o,\, z)}((p)_1).$$

II A N I N T U I T I O N I S T I C P R O O F O F T H E C O M P L E T E N E S S T H E O R E M

In chapter I, section 3, we have sketched two different proofs of the completeness theorem for intuitionistic predicate calculus with respect to the models, defined in chapter I, definition 2.1 and 2.2.

That such a proof can be given in intuitionistic metamathematics is due to the fact, that following an idea of W. Veldman [10], I consider the fan of <u>all</u> models, exploding or non-exploding.

In [10], W. Veldman gave a Henkin-type completeness proof for Kripke models. In chapter I, theorem 3.2 it is indicated how one can adapt this proof to a completeness proof with respect to our semantics, as described in chapter I.

In chapter I, theorem 3.5, 3.6 and 3.7, we also sketched a Gödel-type completeness proof, which was an adaptation of the author's completeness proof for Beth-models ([11]).

In this chapter II we will give this Gödel-type completeness proof in full detail.

For technical details I am indebted to M. Fitting's book [18].

§1 Proof Theory

The proof-system, which is described below, is a reformulation of M. Fitting's system in [18].

Let $\tau : \mathbb{N} \to \mathbb{N}$ be given, with $\tau(o) = 0$.

Suppose the alphabet of our language has the following symbols:

Logical symbols:	$v_1, v_2, \ldots$	free variables
	$x_1, x_2, \ldots$	bound variables
	$\vee, \&, \to$	connectives
	$(x), (Ex)$	quantifiers

Predicate symbols: $P_1, P_2, \ldots$
$f = P_o$ (falsity)

We will use v for arbitrary free variables v_i with $i = 1, 2, \ldots$
x for arbitrary bound variables x_i with $i = 1, 2, \ldots$ and
P and Q for arbitrary predicate symbols P_i with $i \in \mathbb{N}$.

P_i is an m-ary predicate symbol $\underset{D}{\equiv} \tau(i) = m$.
By definition f is o-ary.

Formulas are defined in the usual way.
If A is a formula, we define $-A \underset{D}{\equiv} A \to f$.

We use different symbols for free and bound variables to simplify the quantifier rules.

Notation: $B(v_1,\ldots,v_n)$ is a formula B, such that each free variable, occurring in B, is an element of $\{v_1,\ldots,v_n\}$.

Definition 1.1: A signed formula is an expression of the form T(A) or F(A), where A is a formula.

If it is clear from the context what is meant, we will simply write TA instead of T(A) and FA instead of F(A). For example, instead of T(B & C) we will mostly write TB&C.

M. Fitting [18] attaches intended meanings to TA and FA, namely:

TA : I have a proof of A, and

FA : I do not have a proof of A.

And he explains the notion of derivability in terms of finding a counterexample: a formula A is derivable in intuitionistic predicate calculus if all ways, which may give a counterexample, close.

I prefer a different treatment: In this paper TA and FA do not have an intended meaning, but are only notational conveniences, used for historical reasons. However, we do attach an intended meaning to a finite set of signed formulas:

Definition 1.2: S is a sequent $=_D$ S is a finite set of signed formulas.

Notation: If S is a sequent, S_T is the set of all TA with $TA \in S$.

<u>Intended interpretation</u> of $\{TA_1, \ldots, TA_n, FB_1, \ldots, FB_m\}$:

$$A_1 \,\&\, \ldots \,\&\, A_n \rightarrow B_1 \vee \ldots \vee B_m .$$

The rules below are to be read: if the situation (s) above the line is (are) the case, the situation below the line is also the case.

<u>Axioms</u>: sequents S such that for some formula A

$TA, FA \in S$ or $Tf \in S$.

<u>Rules</u>:

$$T\& \quad \frac{S,\ TA\&B,\ TA,\ TB}{S,\ TA\&B} \qquad\qquad F\& \quad \frac{S, FA \qquad\qquad S, FB}{S,\ FA\&B}$$

$$T\vee \quad \frac{S,\ TA\vee B,\ TA \qquad S,\ TA\vee B,\ TB}{S,\ TA\vee B} \qquad\qquad F\vee \quad \frac{S,\ FA,\ FB}{S,\ FA\vee B}$$

$$T\rightarrow \quad \frac{S,\ TA\rightarrow B,\ FA \qquad S,\ TA\rightarrow B,\ TB}{S,\ TA\rightarrow B} \qquad\qquad F\rightarrow \quad \frac{S_T,\ TA,\ FB}{S,\ FA\rightarrow B}$$

$$TE \quad \frac{S,\ T(Ex)A(x),\ TA(v)}{S,\ T(Ex)A(x)} \qquad\qquad FE \quad \frac{S,\ FA(v)}{S,\ F(Ex)A(x)}$$

v not occurring in S

$$T(\ ) \quad \frac{S,\ T(x)A(x),\ TA(v)}{S,\ T(x)A(x)} \qquad\qquad F(\ ) \quad \frac{S_T,\ FA(v)}{S,\ F(x)A(x)}$$

v not occurring in S

A <u>proof-tree</u> or <u>derivation</u> starts from axioms and proceeds downwards by means of finitely many applications of the rules. The bottom sequent is said to be the conclusion of the proof tree.

Definition 1.3: Let A be a formula and Γ a possibly infinite set of formulas.

A is derivable in IPC $\underset{D}{=}$ there is a proof-tree with $\{FA\}$ as conclusion.

Notation: $\vdash A$.

A is derivable from Γ in IPC $\underset{D}{=}$ there are formulas $B_1, \ldots, B_n$ from Γ such that there is a proof-tree with $\{TB_1, \ldots, TB_n, FA\}$ as conclusion.

Notation: $\Gamma \vdash A$.

Example 1.4: $\vdash (x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

Proof:

$$T(x)(P(x)) \vee Q, T(x)P(x), TP(v_1), FP(v_1), FQ \qquad T(x)(P(x)) \vee Q, TQ, FP(v_1), FQ$$

$$T(x)(P(x)) \vee Q, T(x)P(x), TP(v_1), FP(v_1) \vee Q \qquad T(x)(P(x)) \vee Q, TQ, FP(v_1) \vee Q$$

$$T(x)(P(x)) \vee Q, T(x)P(x), FP(v_1) \vee Q$$

$$T(x)(P(x)) \vee Q, T(x)P(x), F(x)(P(x) \vee Q) \qquad T(x)(P(x)) \vee Q, TQ, F(x)(P(x) \vee Q)$$

$$T(x)(P(x)) \vee Q, F(x)(P(x) \vee Q)$$

$$|$$

$$F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$$

Notation: We use $|$ to denote an application of rule $F\rightarrow$ or $F(\ )$.

Properties of $\vdash$:

1. Replacing in rule $F \rightarrow$ and $F(\ )$, S_T by S, we get a proof-system for classical predicate calculus (CPC) and the notion of $\vdash_{CPC} A$.

   Evidently, if $\vdash A$, then $\vdash_{CPC} A$.

2. Notice, that $\dfrac{S,\ FA(v_1)\ ,\ FA(v_2)}{S, F(Ex)\ A(x)}$ is a derived rule, because $S \cup \{F(Ex)A(x)\} =$ $S \cup \{F(Ex)A(x),\ F(Ex)A(x)\}$.

3. For A a formula of the propositional calculus and for Γ a finite set of formulas of the propositional calculus $\Gamma \vdash A$ is decidable. The decision procedure is afforded by the process of attempting to construct a derivation of A from Γ in IPC.

4. In all F-rules, except in rule $F \rightarrow$, going from bottom to top, only F-formulas are introduced, while in rule $F \rightarrow$ S is replaced by S_T.

   So: if $\{FA_1\ ,\ldots,\ FA_m\}$ is the conclusion of a proof-tree, then it is impossible that in the axioms S, TB, FB of the proof-tree, TB results from FA_i for some i and FB results from FA_j for some j, $j \neq i$, by application of the rules.

This observation gives the following results, which are already known from the literature (see for example [24] and [18] , chapter 6, lemma 4.2 and 4.3):

i) Disjunction Property: If $\vdash B \vee C$, then $\vdash B$ or $\vdash C$.

ii) Explicit Definability Theorem:

   If $\vdash (Ex)\ A(x,\ v_1\ ,\ldots,\ v_m)$, then

$\vdash A(v_k, v_1, \ldots, v_m)$, where v_k is one of the v_i, $1 \leqslant i \leqslant m$, or v_k is a free variable, which does not occur in $(Ex)\,A(x, v_1, \ldots, v_m)$.

Also the following theorem is known from the literature (see [22]), but I want to prove it here again explicitly in intuitionistic metamathematics.

Theorem 1.5: For A a prenex formula, $\vdash A$ is decidable.

Proof: $\quad Z((Ex)B(x, v_1, \ldots, v_m)) \underset{D}{\equiv}$

$\{B(v_k, v_1, \ldots, v_m), B(v_1, v_1, \ldots, v_m), \ldots, B(v_m, v_1, \ldots v_m)\}$ where v_k is the first free variable, not occurring in (Ex) $B(x, v_1, \ldots, v_m)$.

$Z((x)B(x, v_1, \ldots, v_m)) \underset{D}{\equiv} \{B(v_k, v_1, \ldots, v_m)\}$, where v_k is again the first free variable, not occurring in (x) $B(x, v_1, \ldots, v_m)$. Clearly, $\vdash (x)\,B(x, v_1, \ldots, v_m)$ iff $\vdash B(v_k, v_1, \ldots, v_m)$ (*) .

If R is a set of formulas of the form $(Ex)B(x, v_1, \ldots, v_m)$ or $(x)B(x, v_1, \ldots, v_m)$, then $Z(R) \underset{D}{\equiv} \bigcup \{Z(B);\, B \in R\}$.

Clearly, if R is finite, then Z(R) is finite and the number of elements of Z(R) can easily be estimated from the definitions above.

Suppose now that A is a prenex formula $Q^1 x_1 \ldots Q^n x_n B$.

Let $R_1 \underset{D}{\equiv} Z(A)$ and by induction $R_k \underset{D}{\equiv} Z(R_{k-1})$, $k = 2, \ldots, n$.

It follows from 4 ii) (Explicit definability theorem) and (*) that A is derivable in IPC iff R_1 contains at least one derivable formula.

By an easy induction with respect to k, k = 2 ,..., n we find that A is derivable iff R_k contains at least one derivable formula .

Consequently, A is derivable iff R_n contains at least one derivable formula. However, all formulas in R_n are quantifier-free.

And, if C is a quantifier-free formula, " $\vdash$ C" is decidable: the decision procedure is afforded by the process of attempting to construct a derivation of C.

So, because R_n is finite, we can decide by a finite method, whether there is a derivable formula in R_n.

q.e.d.

Let A be a formula and let Γ be a set of formulas. In the next remark we use the expressions " $\Gamma \vdash$ A is decidable" and " Γ is decidable" in the following sense:

$\Gamma \vdash A$ is decidable $\underset{D}{\equiv}$ We can decide in finitely many steps whether $\Gamma \vdash A$ holds or does not hold.

Γ is decidable $\underset{D}{\equiv}$ We have a finite method, such that we can decide for each formula B whether B belongs to Γ or does not belong to Γ .

<u>Remark:</u> For A a formula of the propositional calculus and for Γ an infinite set of formulas of the propositional calculus, $\Gamma \vdash A$ is not decidable, even when Γ is decidable.

Weak counterexample: $F(i) \underset{D}{=} i_1^{\,i_0+2} + i_2^{\,i_0+2} = i_3^{\,i_0+2}$.

$F \underset{D}{=} (Ei)(i \in \mathbb{N}\ \&\ F\ (i))$.

Let P and Q be propositional variables.

$\Gamma \underset{D}{=} \{C_1, C_2, C_3, \ldots\}$ where

$$C_i \underset{D}{=} \underbrace{P \rightarrow (\ldots \rightarrow P)}_{i \text{ times}} \quad \text{if not } F(i)$$

and $C_i \underset{D}{=} \underbrace{(P \rightarrow (\ldots \rightarrow P))}_{(i-1) \text{ times}} \rightarrow Q$ if for some $j \leqslant i$, $F(j)$.

Then: if Γ is finite, then $-F\ \&\ --F$.

So Γ is not finite and it is also easy to see that Γ is decidable.

Now: $\Gamma \vdash Q$ iff for some i, F(i).

So, a decision about $\Gamma \vdash Q$ would be a decision about the existence of some natural number i such that F(i).

§2. Systematic Procedure for Searching Derivations.

Let Δ be a set of signed formulas and let P be a set of free individual variables.

Definition 2.1: Δ is a Hintikka element with respect to P $\underset{D}{=}$

1. If $TB\&C \in \Delta$, then $TB \in \Delta$ and $TC \in \Delta$.

2. If $TB\vee C \in \Delta$, then $TB \in \Delta$ or $TC \in \Delta$.

3. If $TB\to C \in \Delta$, then $FB \in \Delta$ or $TC \in \Delta$.

4. If $FB\&C \in \Delta$, then $FB \in \Delta$ or $FC \in \Delta$.

5. If $FB\vee C \in \Delta$, then $FB \in \Delta$ and $FC \in \Delta$.

6. If $T(x)B(x) \in \Delta$, then for all $v \in P$, $TB(v) \in \Delta$.

7. If $F(Ex)B(x) \in \Delta$, then for all $v \in P$, $FB(v) \in \Delta$.

It is easy to see, that if Δ' is a set of signed formulas and P is a set of free variables, then we can extend Δ' to a Hintikka element Δ with respect to P.

Remark: The expression "Δ is a Hintikka element with respect to P" is due to M. Fitting ([18], page 57 and 32).

Notation: If Δ is a set of signed formulas, FVar (Δ) is the set of all free variables, occurring in Δ.

Let us say we apply an inverted rule to a set S of signed formulas, if we replace S by one of the possible premisses, S', S'' according to a rule of the system, i.e. $\frac{S',\ S''}{S}$ would be an application of one of the rules.

By a <u>split formula</u> we mean a formula of the form $FB \to C$ or $F(x)B(x)$.

The expression "split-formula" is due to A.S. Troelstra [16] and can be explained as follows:
Suppose we would systematically search for derivations of a sequent S, by applying inverted rules to see whether we end up with a set of axioms.
Now, an application of e.g. $F \to$ may destroy possibilities; e.g. if we consider

$$S, FA \to B, FC \to D$$

there are two possibilities for applying $F \to$ inversely, yielding

$$S_T , TA , FB \quad \text{and} \quad S_T , TC , FD$$

respectively; but in the first case we have lost the possibility of treating $FC \to D$, in the second case of treating $FA \to B$. We have to pursue the possibilities of finding a proof tree for S_T, TA , FB and S_T , TC , FD ; it is sufficient to find a proof tree in one of these cases.

Let A be a formula.

<u>Definition 2.2</u>; <u>The systematic procedure for searching a derivation of A in IPC</u> is the procedure, given by the following description:

Step O: Consider{ FA } .

Apply all inverted rules, except the rules F $\rightarrow$, F( ), TE, as many times as possible, with the restriction, that we apply the inverted rule T( ) and FE only with respect to the free variables occurring in {FA} .

Having finished step O of our procedure, we have one (if T V, F&, and T $\rightarrow$ have not been applied) or more Hintikka elements Γ_o^i with respect to F Var (Γ_o^i) Each Γ_o^i is finite.

If Γ_o^i contains TB and FB for some formula B, then we add Tf to Γ_o^i .

Motivation for step 1: Each Γ_o^i may contain several split formulas, so for each Γ_o^i there may be several ways to go on. Each way may result in an axiom, so for each Γ_o^i we have to consider all possibilities to go on (step 1 a).

In addition we can apply the inverted rules T( ), FE also with respect to free variables not occurring in Γ_o^i (step 1 b).

Observe also, that in step O we did not apply the inverted rule TE.

Step 1 a: For each split formula in Γ_o^i , say the m th, we form one or more successors $\Gamma_{o,m}^{i,j}$ of Γ_o^i as follows:

i) Let the m^{th} split formula in Γ_o^i be of the form FB $\rightarrow$ C. Consider $(\Gamma_o^i)_T$, TB , FC and assume $(\Gamma_o^i)_T$, TB, FC contains k formulas of the form T(Ex)D(x), say T(Ex)D_1(x) ,..., T(Ex)D_k(x).

Suppose $b_1, \ldots, b_k$ are the first k free variables, which do not occur in $(\Gamma^i_o)_T$, TB , FC.

Form $\Delta = (\Gamma^i_o)_T$, TB , FC , $TD_1(b_1), \ldots, TD_k(b_k)$.
Apply all inverted rules, except F $\rightarrow$, F( ), and TE as many times as possible to Δ , with the restriction that we apply T( ) and FE only with respect to FVar $(\Delta) \cup \{ b_1 \}$.

Having finished step 1 a i) we have one (if T $\vee$, F& and T $\rightarrow$ have not been applied) or more Hintikka-elements $\Gamma^{i,j}_{o,m}$ with respect to FVar $(\Gamma^{i,j}_{o,m})$.
Each $\Gamma^{i,j}_{o,m}$ is finite.

If $\Gamma^{i,j}_{o,m}$ contains TB and FB for some formula B, then we add Tf to $\Gamma^{i,j}_{o,m}$.

ii) Let the m^{th} split formula in Γ^i_o be of the form F(x)B(x).
Consider $(\Gamma^i_o)_T$, FB(b), where b is the first free variable not occurring in $(\Gamma^i_o)_T$.
Now proceed like in i).

If Γ^i_o contains l split formulas, i) and ii) yield successors $\Gamma^{i,j}_{o,1}, \ldots, \Gamma^{i,j}_{o,l}$ of Γ^i_o .

The other successors $\Gamma^{i,j}_{o,l+1}$, $\Gamma^{i,j}_{o,l+2}, \ldots$ are the same and are yielded by step 1 b:

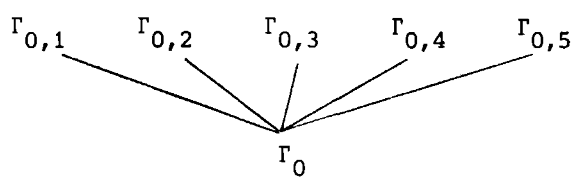
<u>Step 1 b</u>: Suppose Γ_o^i contains k formulas of the form $T(Ex)D(x)$, say $T(Ex)D_1(x), \ldots, T(Ex)D_k(x)$.
Suppose $b_1, \ldots, b_k$ are the first free variables, not occurring in Γ_o^i.
Form $\Delta = \Gamma_o^i$, $T\,D_1(b_1), \ldots, TD_k(b_k)$.
Apply all inverted rules, except $F \rightarrow$, $F(\ )$, and TE, as many times as possible to Δ , with the restriction that we apply $T(\ )$ and FE only with respect to $FVar(\Delta) \cup \{b_1\}$.

$\Gamma_{o,\,l+1}^{i,\,j}$ is a Hintikka-element with respect to FVar

$(\Gamma_{o,\,l+1}^{i,\,j})$, and $\Gamma_{o,\,l+1}^{i,\,j}$ is finite.

If $\Gamma_{o,\,l+1}^{i,\,j}$ contains TB and FB for some formula B, then we add Tf to $\Gamma_{o,\,l+1}^{i,\,j}$.

Having finished step 1 of our procedure, we have one (if T v, F& and T $\rightarrow$ have nowhere been applied) or more partial trees of the form

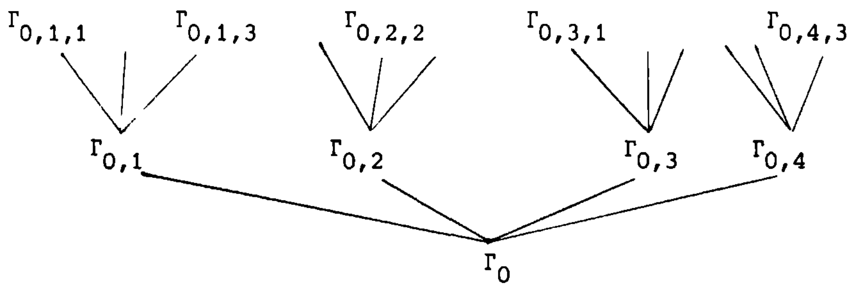


where each $\Gamma_{o,i}$ is a Hintikka-element with respect to $FVar(\Gamma_{o,i})$ and each $\Gamma_{o,i}$ is finite.

<u>Step 2</u> simply repeats this process, giving successors

$$\Gamma_{o,\,m,\,n}^{i,\,j,\,k} \, .$$

Having finished step 2 of our procedure, we have one (if T ∨, F&, T → have nowhere been applied) or more partial trees of the form:



The other steps of our procedure are similar.

Definition 2.3: We call the trees, which we get by our systematic procedure for searching a derivation of A in IPC, search trees for A, or search trees starting with FA.

Notice, there may be many search trees for A, because of the rules T ∨, F&, T → .

Example 2.4: We apply our systematic procedure for searching a derivation of (x) (P(x) ∨ Q) → (x)(P(x)) ∨ Q in IPC.

Step 0 yields: F(x)(P(x) ∨ Q) → (x)(P(x)) ∨ Q.

Step 1 yields two partial trees, namely

I $TP(v_1)$

$TP(v_1) \vee Q$, F(x)P(x), FQ

T(x)(P(x) ∨ Q), F(x)(P(x)) ∨ Q F(x)(P(x) ∨ Q) →(x)(P(x)) ∨ Q

F(x)(P(x) ∨ Q) → (x)(P(x)) ∨ Q

II

TQ

$TP(v_1) \vee Q,\ F(x)P(x),\ FQ$

$T(x)(P(x) \vee Q),\ F(x)(P(x)) \vee Q \qquad F(x)(P(x) \vee Q) \rightarrow (x)(P(x)) \vee Q$

$F(x)(P(x) \vee Q) \rightarrow (x)(P(x)) \vee Q$

Step 2 yields many more partial trees.

One of the partial trees, corresponding with I is:

$TP(v_1)\ ,\ TQ$

$TP(v_1) \vee Q,\ TP(v_2) \vee Q$

$T(x)(P(x) \vee Q),\ FP(v_2)$

$TP(v_2)$

$TP(v_2) \vee Q$

$TP(v_1)$

$TP(v_1) \vee Q,\ F(x)P(x),\ FQ$

$T(x)(P(x) \vee Q),\ F(x)(P(x)) \vee Q$

$\Gamma_{0,2,1} \qquad \Gamma_{0,2,2}$

$TP(v_1)$

$TP(v_1) \vee Q,\ F(x)P(x),\ FQ$

$T(x)(P(x) \vee Q),\ F(x)(P(x)) \vee Q \qquad \Gamma_{0,2}$

$F(x)(P(x) \vee Q) \rightarrow (x)(P(x)) \vee Q$

Where $\Gamma_{0,2}$ is $F(x)(P(x) \vee Q) \rightarrow (x)(P(x)) \vee Q$,

$\Gamma_{0,2,1}$ is $TP(v_1)$

$TP(v_1) \vee Q,\ F(x)(P(x)),\ FQ$

$T(x)(P(x) \vee Q)\ ,\ F(x)(P(x)) \vee Q$

and $\Gamma_{0,2,2}$ is $F(x)(P(x) \vee Q) \rightarrow (x)(P(x)) \vee Q$.

<u>Example 2.5</u>: We apply our systematic procedure for searching a derivation of $(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$ in IPC.

Step 0 yields: $F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

Step 1 yields two partial trees, namely

I

$TP(v_1)$

$T(x)P(x)$

$T(x)(P(x)) \vee Q,\ F(x)(P(x) \vee Q)$ $\qquad$ $F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

$F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

II

TQ

$T(x)(P(x)) \vee Q,\ F(x)(P(x) \vee Q)$ $\qquad$ $F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

$F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

Applying step 2 we get again two partial trees, namely one corresponding with I and the second corresponding with II:

I'

Tf

$TP(v_2)$

$TP(v_1)$

$T(x)P(x),\ FP(v_2),\ FQ$

$T(x)(P(x)) \vee Q,\ FP(v_2) \vee Q$

$TP(v_1)$ $\qquad$ $F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

$T(x)P(x)$

$T(x)(P(x)) \vee Q,\ F(x)(P(x) \vee Q)$

$F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

II'

Tf

$TQ,\ FP(v_1)\ ,\ FQ$

$T(x)(P(x)) \vee Q,\ FP(v_1) \vee Q$

TQ

$T(x)(P(x)) \vee Q,\ F(x)(P(x) \vee Q) \qquad F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

$F(x)(P(x)) \vee Q \rightarrow (x)(P(x) \vee Q)$

Notice, we have found now a derivation of $(x)(P(x)) \vee Q \rightarrow$ $(x)(P(x) \vee Q)$ in IPC, namely the following:

$T(x)(Px) \vee Q,\ T(x)Px,\ TPv_2\ ,\ FPv_2\ ,FQ$

$T(x)(Px) \vee Q,\ T(x)Px,\ TPv_2\ ,\ FPv_2 \vee Q \qquad T(x)(Px) \vee Q,\ TQ,\ FPv_1\ ,\ FQ$

$T(x)(Px) \vee Q,\ T(x)Px,\ FPv_2 \vee Q \qquad T(x)(Px) \vee Q,\ TQ,\ FPv_1 \vee Q$

$T(x)(Px) \vee Q,\ T(x)Px,\ F(x)(Px \vee Q) \qquad T(x)(Px) \vee Q,\ TQ,\ F(x)(Px \vee Q)$

$T(x)(Px) \vee Q,\ F(x)(Px \vee Q)$

$F(x)(Px) \vee Q \rightarrow (x)(Px \vee Q)$

<u>Remark 2.6</u>: From the description of the systematic procedure for searching a derivation of A in IPC, it will be clear, that we can give to all our search trees the tree structure of $\bigcup_n \mathbb{N}^n$.

We can also modify the search procedure in an appropriate way, such that we give to all our search trees the tree structure of $\bigcup_n \{0,1\}^n$.

Notation: SEQ is by definition $\bigcup_n \mathbb{N}^n$.

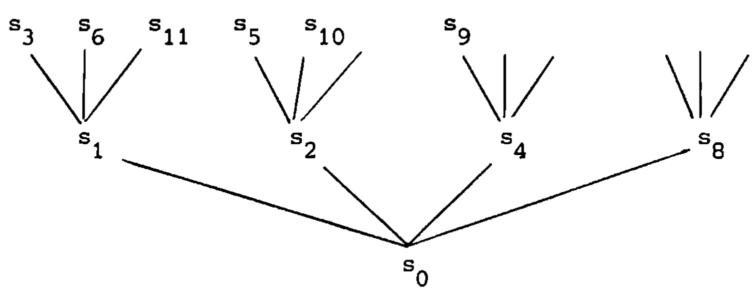
Definition 2.7: A search tree for A is closed iff for some $s \in$ SEQ, Tf occurs at the node s.

We will show now, that if all search trees for A are closed, then $\vdash$ A.

Therefore we need the following

Lemma 2.8: The collection of all search trees for A is a fan F_A.

Proof: Let s_0 , s_1 , s_2 ,... be a fixed enumeration of SEQ , such that if s_i is an initial segment of s_j , then $i \leqslant j$.
For example:



Since each set of signed formulas, associated with an $s \in$ SEQ, is finite, we can identify a search tree for A with a sequence n_0 , n_1 , n_2 ,..., of natural numbers, where - see the picture -

n_0 is the Gödel-number of a result of applying our systematic procedure to {FA} .

n_1 is the Gödel-number of a result of applying our systematic procedure to n_0

n_2 is ... ... n_0
n_3 is ... ... n_1
n_4 is ... ... n_0
n_5 is ... ... n_2

From this we see that the search trees for A form a fan F_A . q.e.d.

<u>Theorem 2.9</u>: If all search trees for A are closed, then $\vdash A$.

More explicitly: If for all $t \in F_A$ there is some natural number n such that Tf occurs in $t(n)$, then $\vdash A$.

Proof: Suppose for all $t \in F_A$ there is some $n \in \mathbb{N}$ such that Tf occurs in $t(n)$.
Then, we know by the fan-theorem:
There is some natural number, say N, such that for all $t \in F_A$ there is some n, $n \leqslant N$, such that Tf occurs in $t(n)$.
Because F_A is a fan, there are only finitely many different $\bar{t}N$ with $t \in F_A$, say $\bar{t}_1 N, \ldots, \bar{t}_k N$.
Together they form a derivation of A.
Hence $\vdash A$.

q.e.d.

§3 Soundness and Completeness.

In this section we want to establish the soundness and the completeness of the formal system, described in section 1, of this chapter II, with respect to the semantics of chapter I, section 2 (definition 2.1 , 2.2 and 2.8). The completeness proof, which we will give here, is a detailed elaboration of the sketch we gave in chapter I, section 3, theorem 3.5, 3.6 and 3.7, in which we made use of the author's completeness proof with respect to Beth-models [11] .

Let A be a formula.

<u>Theorem 3.1 (Soundness)</u>: If $\vdash A$, then $\models A$.

Proof: By induction on the length of a derivation of A in IPC.

In order to establish the completeness theorem we wish to show that any search tree for A can be interpreted as a model $M = \langle S_M, T_M \rangle$.

<u>Lemma 3.2</u>: Each search tree for A has the following property: To each $s \in SEQ$ is associated a set of signed formulas, which we will also denote with s , such that for each $s \in SEQ$:

1. If there is a formula B, such that TB and FB both occur in s , then Tf occurs in s.

2. Each s is a Hintikka element (see definition 2.1) with respect to the set of free variables, which occur in the signed formulas of s.

3. If TA occurs in s , then for all s' ϵ SEQ, if s is an initial segment of s', then TA also occurs in s'.

4. If FB(v_1 ,..., v_m) occurs in s , then there is some infinitely proceeding sequence α through s , such that for all k ϵ IN, if k $\geqslant$ length (s), then FB (v_1 ,..., v_m) occurs in $\bar{\alpha}k$.

5. If FB $\vee$ C occurs in s , then there is some infinitely proceeding sequence α through s , such that for all k $\geqslant$ length (s) both FB and FC occur in $\bar{\alpha}k$.

6. If F(Ex)B(x) occurs in s , then there is some infinitely proceeding sequence α through s , such that for all natural numbers i there is some k such that for all l $\geqslant$ k FB (v_i) occurs in $\bar{\alpha}l$.

7. If T(Ex)B(x) occurs in s , then for all infinitely proceeding sequences α through s there is some natural number i and some k , such that TB (v_i) occurs in $\bar{\alpha}k$.

8. If T(x)B(x) occurs in s , then for all infinitely proceeding sequences α through s and for all natural numbers i there is some k , such that TB(v_i) occurs in $\bar{\alpha}k$.

9. If FB$\rightarrow$C occurs in s , then there is an s' ϵ SEQ, such that s is an initial segment of s' , and such that both TB and FC occur in s' .

10. If F(x)B(x) occurs in s , then there is an s'ϵ SEQ and a natural number i, such that s is an initial segment of s' and FB(v_i) occurs in s'.

Proof: Immediate from the description of the systematic procedure for searching a derivation of A in IPC; the infinitely proceeding sequence α, of which the existence is claimed in 4,5 and 6, is obtained by a succession of b-steps in the search procedure (definition 2.2).

q.e.d.

Definition 3.3 (Interpretation of a search tree as a model):

For t a search tree for A, $(t \in F_A)$, define $M(t) = < S_{M(t)}, T_{M(t)} >$ as follows:

i) $S_{M(t)}$ is the spread law, which allows just those sequences, which are in the search tree t.
By remark 2.6 $S_{M(t)}$ can be taken fixed for all $t \in F_A$, namely the universal spread law (or the binary spread law).

ii) For $< a_0, \ldots, a_k >$ allowed by the spread law $S_{M(t)}$,
$T_{M(t)}(< a_0, \ldots, a_k >)(<P, n_1, \ldots, n_m>) = 1$
$\Leftrightarrow$ TP $(v_{n_1}, \ldots, v_{n_m})$ occurs in $<a_o, \ldots, a_k>$.

Then by 3 in lemma 3.2, if
$T_{M(t)}(<a_0, \ldots, a_k>)(<P, n_1, \ldots, n_m>) = 1$, then also
$T_{M(t)}(<a_0, \ldots, a_k, a_{k+1}>)(<P, n_1, \ldots n_m>) = 1$.
Hence we have

Lemma 3.4: For t a search tree for A $(t \in F_A)$, M(t), as defined in 3.3, is a model in the sense of definition 2.1 of chapter I.

Now, the completeness theorem will follow from theorem 2.9 and the following

Theorem 3.5: Let $t \in F_A$ be a search tree for A and let $M = M(t)$ be the corresponding model (definition 3.3). Then for all $s \in SEQ$ and for all formulas $B(v_{n_1}, \ldots, v_{n_m})$:

I. If $TB(v_{n_1}, \ldots, v_{n_m})$ occurs in s, then for all infinitely proceeding sequences α through s, $M_\alpha \models' B(v_{n_1}, \ldots, v_{n_m})[n_1, \ldots, n_m]$. (see definition 2.12 of chapter I)

II. If $FB(v_{n_1}, \ldots, v_{n_m})$ occurs in s and for all infinitely proceeding sequences α through s, $M_\alpha \models' B(v_{n_1}, \ldots, v_{n_m})[n_1, \ldots, n_m]$, then M explodes.

Proof: By induction on the complexity of B.

1. Suppose $TP(v_{n_1}, \ldots, v_{n_m})$ occurs in s.
   Then for each α through s there is some k such that $TP(v_{n_1}, \ldots, v_{n_m})$ occurs in $\bar{\alpha}k$.
   Hence for each α through s there is some k such that $T_M(\bar{\alpha}k)(\langle P, n_1, \ldots, n_m\rangle) = 1$.
   Hence for all α through s,
   $M_\alpha \models' P(v_{n_1}, \ldots, v_{n_m})[n_1, \ldots, n_m]$.

Suppose $FP(v_{n_1}, \ldots, v_{n_m})$ occurs in s (i) and for all α through s, $M_\alpha \models' P(v_{n_1}, \ldots, v_{n_m})[n_1, \ldots, n_m]$ (ii).

From i it follows by 4 of lemma 3.2 that there is some α through s such that for all $k \geqslant \text{length}(s)$ $FP(v_{n_1},\ldots,v_{n_m})$ occurs in $\bar{\alpha}k$.

For this α it follows from ii that either M explodes or for some l, $TP(v_{n_1},\ldots, v_{n_m})$ occurs in $\bar{\alpha}l$.

In both cases it follows that M explodes.

2. B = C & D : Use 2 of lemma 3.2.

3. $B = C \vee D$: Use, to establish I, 2 of lemma 3.2.

To establish II, suppose $FC \vee D$ occurs in s (i) and for all α through s $M_\alpha \models' C \vee D$ (ii)

From i it follows by 5 of lemma 3.2 that there is some α through s, such that for all $k \geqslant$ length (s) both FC and FD occur in $\bar{\alpha}k$.

For this α we know by ii that $M_\alpha \models' C$ or $M_\alpha \models' D$.

Suppose $M_\alpha \models' C$. Then by lemma I, 2.13 it follows that there is some l such that for all β, if $\bar{\beta}l = \bar{\alpha}l$, then $M_\beta \models' C$.

Now FC occurs in $\bar{\alpha}l$ and for all β through $\bar{\alpha}l$, $M_\beta \models' C$.

Hence, by induction hypothesis, M explodes.

4. $B = C \to D$

Suppose $TC \to D$ occurs in s.

We have to show that for all α through s, $M_\alpha \models' C \to D$, i.e. for all α through s there is some $k \in \mathbb{N}$, such that for all β, if $\bar{\beta}k = \bar{\alpha}k$ and $M_\beta \models' C$, then $M_\beta \models' D$.

To each α through s take $k \underset{D}{=}$ length (s).

Suppose now, that $\bar{\beta}k = \bar{\alpha}k$ and $M_\beta \models' C$.

Then by I, 2.13 it follows that there is some $l \in IN$, $l \geq k$ such that for all γ, if $\bar{\gamma}l = \bar{\beta}l$, then $M_\gamma \overset{|}{\models} C$.

By 3 of lemma 3.2 $TC \to D$ occurs in $\bar{\beta}l$ and by 2 of lemma 3.2 either FC or TD occurs in $\bar{\beta}l$.

If TD occurs in $\bar{\beta}l$, then by induction hypothesis $M_\beta \overset{|}{\models} D$.

If FC occurs in $\bar{\beta}l$, then, because for all γ through $\bar{\beta}l$, $M_\gamma \overset{|}{\models} C$, by the induction hypothesis, M explodes and hence also $M_\beta \overset{|}{\models} D$.

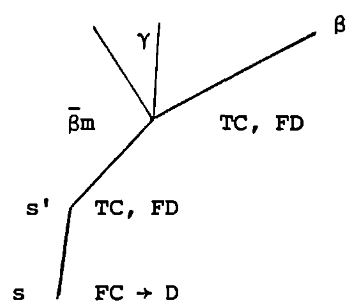
Suppose $FC \to D$ occurs in s (i) and for all α through s, $M_\alpha \overset{|}{\models} C \to D$ (ii).

From i it follows by 9 of lemma 3.2 that there is some $s' \in SEQ$, such that s is an initial segment of s' and such that both TC and FD occur in s'.

By 3 and 4 of lemma 3.2 there is some β through s' such that for all $l \geq$ length (s') both TC and FD occur in $\bar{\beta}l$.

By induction hypothesis $M_\beta \overset{|}{\models} C$, and hence by ii, $M_\beta \overset{|}{\models} D$; Hence by I, 2.13 there is some m such that for all γ through $\bar{\beta}m$, $M_\gamma \overset{|}{\models} D$.

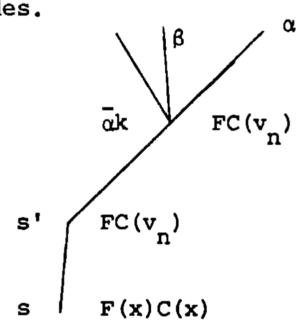
Now FD occurs in $\bar{\beta}m$ and for all γ through $\bar{\beta}m$, $M_\gamma \overset{|}{\models} D$; so from the induction hypothesis it follows that M explodes.



5 $B = (x)C(x)$

Suppose $T(x)C(x)$ occurs in s.
By 8 of lemma 3.2, for all α through s and for all natural numbers n there is some k such that $TC(v_n)$ occurs in $\bar{\alpha}k$. Hence, by induction hypothesis, for all α through s and for all $n \in \mathbb{N}$, $M_\alpha \stackrel{\prime}{\models} C(v)\ [n]$.
Hence for all α through s, $M_\alpha \stackrel{\prime}{\models} (x)\ C(x)$.

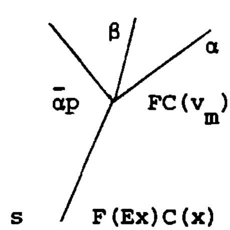
Suppose $F(x)C(x)$ occurs in s (i) and for all α through s, $M_\alpha \stackrel{\prime}{\models} (x)C(x)$ (ii).
From i it follows by 10 of lemma 3.2 that there is an $s' \in SEQ$ and a natural number n, such that s is an initial segment of s' and such that $FC(v_n)$ occurs in s'.
By 4 of lemma 3.2 there is some α through s' such that for all $l \geqslant$ length (s'), $FC\ (v_n)$ occurs in $\bar{\alpha}l$.
For this α there is by ii some k such that for all natural numbers m and for all β through $\bar{\alpha}k$,
$M_\beta \stackrel{\prime}{\models} C(v_m)[m]$.
So $FC(v_n)$ occurs in $\bar{\alpha}k$ and for all β through $\bar{\alpha}k$,
$M_\beta \stackrel{\prime}{\models} C(v_n)\ [n]$. Hence by the induction hypothesis,
M explodes.



6. $B = (Ex)C(x)$

Suppose $T(Ex)C(x)$ occurs in s.
Then by 7 of lemma 3.2 it follows that for all α through s there is some $n \in \mathbb{N}$ and some k such that $TC(v_n)$ occurs in $\bar{\alpha}k$.
Hence, by the induction hypothesis, for all α through s there is some $n \in \mathbb{N}$ such that $M_\alpha \models' C(v_n)[n]$.
Hence for all α through s , $M_\alpha \models' (Ex)C(x)$.

Suppose $F(Ex)C(x)$ occurs in s (i) and for all α through s, $M_\alpha \models' (Ex)C(x)$ (ii).
From i it follows by 6 of lemma 3.2 that there is some α through s , such that for all $n \in \mathbb{N}$ there is some $k(n)$ such that for all $l \geq k(n)$ $FC(v_n)$ occurs in $\bar{\alpha}l$.
For this α it follows from ii that there is some $m \in \mathbb{N}$ such that $M_\alpha \models' C(v_m)\ [m]$.
By lemma I, 2.13, for this α there is some $m \in \mathbb{N}$ and some $p \geq k(m)$ such that for all β through $\bar{\alpha}p$, $M_\beta \models' C(v_m)\ [m]$.
Now $FC(v_m)$ occurs in $\bar{\alpha}p$ and for all β through $\bar{\alpha}p$, $M_\beta \models' C(v_m)\ [m]$.
Hence it follows from the induction hypothesis that M explodes.



q.e.d.

<u>Theorem 3.6</u> (Completeness of the formal system of II, section 1 with respect to the semantics of I, section 2):
If $\models A(v_{n_1}, \ldots, v_{n_m})$, then $\vdash A(v_{n_1}, \ldots, v_{n_m})$.

Proof: Suppose $\models A(v_{n_1}, \ldots, v_{n_m})$.

Then by I, 3.6 $\models' A(v_{n_1}, \ldots, v_{n_m})$ (definition 2.12 of chapter I).

Hence for all search trees $t \in F_A$, $M(t) \models' A(v_{n_1}, \ldots, v_{n_m})$, while $FA(v_{n_1}, \ldots, v_{n_m})$ occurs in the bottom node s_o of all trees $t \in F_A$. So, by theorem 3.5, for all search trees $t \in F_A$, $M(t)$ explodes, i.e. for all search trees $t \in F_A$ there is some $n \in IN$ such that Tf occurs in $t(n)$. In other words: all search trees for $A(v_{n_1}, \ldots, v_{n_m})$ are closed.
Hence, by theorem 2.9, $\vdash A(v_{n_1}, \ldots, v_{n_m})$.

q.e.d.

Let $E_1, \ldots, E_n$ and A be formulas.
If we replace in step 0 of the systematic procedure for searching a derivation of A in IPC (definition 2.2) $\{FA\}$ by $\{TE_1, \ldots, TE_n, FA\}$, then we get a systematic procedure for searching a derivation of A from $E_1, \ldots, E_n$ in IPC.
Hence the proof of theorem 3.6 can be generalized to a proof of

<u>Theorem 3.7</u>: If $E_1, \ldots, E_n \models A$ (I, definition 2.8), then $E_1, \ldots, E_n \vdash A$ (II, definition 1.3).

For Γ a countably infinite sequence of formulas we can even modify the systematic procedure of definition 2.2 to a systematic procedure for searching a derivation of A from Γ in IPC. Hence we can generalize theorem 3.6 to a <u>strong completeness</u> theorem:

<u>Theorem 3.8</u>: Let Γ be a countably infinite sequence of formulas.
If $\Gamma \models A$ (I,definition 2.8), then $\Gamma \vdash A$ (II, definition 1.3).

<u>Remark 3.9</u>: If we start the systematic procedure, described in definition 2.2 with {TC} instead of with {FA} , then we get a fan (lemma 2.8) of models (lemma 3.4) of C (theorem 3.5).
More precisely, for each model M in the fan, $M \models' C$
(we do not have $M \models C$!).

<u>Remark 3.10</u>: As pointed out by A. Troelstra in [16] , we have, as a consequence of the completeness proof in theorem 3.6, obtained an intuitionistic model-theoretic proof of the fact that our formal system of section 1 is closed under cut, where cut is the rule:

$$\text{Cut}\ \frac{TB_1, \ldots, TB_k, FA \qquad TA, TC_1, \ldots, TC_m, FD_1, \ldots, FD_n}{TB_1, \ldots, TB_k, TC_1, \ldots TC_m, FD_1, \ldots, FD_n}$$

To see this, note that the derivability of

$S = \{ TA_1 ,\ldots, TA_n , FB_1 ,\ldots, FB_m\}$ is equivalent to:

for each model M (I, definition 2.1) and for each admissible infinitely proceeding sequence α , if $M_\alpha \models A_1 \& \ldots \& A_n$, then $M_\alpha \models B_1 \vee\ldots\vee B_m$.

Assume $\{TB_1 ,\ldots, TB_k , FA\}$ and $\{TA , TC_1 ,\ldots, TC_m ,$ $FD_1 ,\ldots, FD_n\}$ to be derivable. (*)

Now consider any model M and any admissible infinitely proceeding sequence α.

If $M_\alpha \models B_1 \&\ldots\& B_k \& C_1 \&\ldots\& C_m$, then, by (*) $M_\alpha \models A$ and hence, again by (*) , $M_\alpha \models D_1 \vee\ldots\vee D_n$.

It follows that $\{TB_1 ,\ldots, TB_k , TC_1 ,\ldots, TC_m ,$ $FD_1 ,\ldots, FD_n \}$ is derivable.

III FIRST STEPS IN INTUITIONISTIC MODELTHEORY.

In this chapter we will do some modeltheory with respect to the models, defined in chapter I, and, as in chapter I and II, we will work again in intuitionistic metamathematics.

In this chapter we will only consider models $M = \langle S, T_M \rangle$, where S is one fixed spreadlaw for all models M , namely the universal spreadlaw. That we can restrict ourselves to this class of models is a consequence of the completeness proof in chapter II (see remark 2.6 , chapter II).

The main tools in this chapter will be two model-constructions:

i) In section 1 we will consider, under a certain condition $C(M_o, M, s)$, the construction of a model $R(M_o, M, s)$ from two models M_o and M with respect to the finite sequence s.

ii) In section 2 we will construct from an infinite sequence $M_o, M_1, M_2, \ldots$ of models a new model $\sum_{i \in IN} M_i$.

Syntactic proofs of the disjunction property and the explicit definability theorem are wellknown (see for example chapter II, section 1). C. Smorynski [17] gave semantic proofs of these theorems with respect to Kripke models, however using classical metamathematics. In section 1 we will give intuitionistically correct, semantic, proofs with respect to the models, defined in chapter I, using Brouwer's Continuity Principle.

Let W be the fan of all models(see chapter I, theorem 2.7)and let Γ be a countably infinite sequence of sentences.

We will prove in section 2:

i) $(A)_{A\in\Gamma}\ (EM)_{M\in W}(\text{not } M \models A)$

$\Leftrightarrow \quad (EM)_{M\in W}(A)_{A\in\Gamma}\ (\text{not } M \models A)$.

ii) $(M)_{M\in W}\ (EA)_{A\in\Gamma}\ (M \models A)$

$\Leftrightarrow \quad (EA)_{A\in\Gamma}\ (M)_{M\in W}\ (M \models A)$.

ii) has already been proved in I, corollary 3.4. But in this chapter we will give another proof, using Brouwer's Continuity Principle and purely semantic arguments.

In I we have already shown two versions of the compactness theorem (corollary 3.3 and 3.8). In section 2 we will also prove some other versions.

The independence of the intuitionistic connectives $\vee$, $\&$, $\rightarrow$ and $-$ will be studied in section 3, using results from section 1.

§1. Submodels; Deduction theorem; The model $R(M_o, M, s)$; Disjunction property; Explicit definability theorem.

Let s and s' be finite sequences of natural numbers.

Definition 1.1: $s \; R \; s' \equiv_D s'$ is a successor of s, i.e. there are natural numbers n, $a_1, \ldots, a_n$, m and $b_1, \ldots, b_m$ such that $s = \langle a_1, \ldots, a_n \rangle$ and $s' = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$.

If $s = \langle a_1, \ldots, a_n \rangle$ and $s' = \langle b_1, \ldots, b_m \rangle$, then $s * s' \equiv_D \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$.

If $s = \langle a_1, \ldots, a_n \rangle$ and $\beta = \langle b_1, b_2, \ldots \rangle$, then $s * \beta \equiv_D \langle a_1, \ldots, a_n, b_1, b_2, \ldots \rangle$.

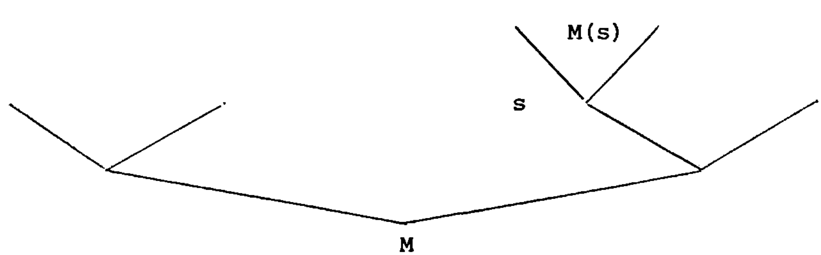
$\langle \; \rangle$ is the empty sequence.

Definition 1.2: Let $M = \langle S, T_M \rangle$ be a model and let s be a finite sequence.

$M(s) = \langle S, T_{M(s)} \rangle$ is the submodel of M, defined by

$T_{M(s)}(s')(\langle P, n_1, \ldots, n_m \rangle) \equiv_D T_M(s*s')(\langle P, n_1, \ldots, n_m \rangle)$

for each m-ary predicate symbol P, for each m-tuple $\langle n_1, \ldots, n_m \rangle$ of natural numbers and for each $m \in \mathbb{N}$.



Lemma 1.3: Let $M(s)$ be a submodel of M.

Then for all admissible α, for all formulas A and for all $a \in \mathbb{N}^{\infty}$

$$M(s)_{\alpha} \models A\,[a] \text{ iff } M_{s*\alpha} \models A\,[a]$$

Proof: by induction on the complexity of A.

Theorem 1.4 (Deduction theorem): Let A and B be formulas. The following propositions are equivalent:

i) $\models A \to B$

ii) $A \models B$

iii) For all models $M \in W$ and for all $a \in \mathbb{N}^{\infty}$, if $M \models A\,[a]$ then $M \models B\,[a]$.

Proof: $M \models A \to B \Leftrightarrow$ for all α and for all $a \in \mathbb{N}^{\infty}$, if $M_{\alpha} \models A\,[a]$, then $M_{\alpha} \models B\,[a]$.
From this it easily follows that $\models A \to B$ and $A \models B$ are equivalent.
Also ii) $\to$ iii) is immediate.
iii) $\to$ ii) : Suppose iii). We have to show that for all models $M \in W$, for all admissible α and for all $a \in \mathbb{N}^{\infty}$, if $M_{\alpha} \models A\,[a]$, then $M_{\alpha} \models B\,[a]$.
So suppose $M_{\alpha} \models A\,[a]$.
Then by lemma I, 1.4 there is some $k \in \mathbb{N}$ such that for all admissible β, if $\bar{\beta}k = \bar{\alpha}k$, then $M_{\beta} \models A\,[a]$.
So $M(\bar{\alpha}k) \models A\,[a]$ (see definition 1.2). Hence by iii) $M(\bar{\alpha}k) \models B\,[a]$. So (lemma 1.3) for all β through $\bar{\alpha}k$, $M_{\beta} \models B\,[a]$. In particular, $M_{\alpha} \models B\,[a]$.

q.e.d.

Let $M_o = \langle S, T_{M_o} \rangle$ and $M = \langle S, T_M \rangle$ be models and let s be a finite sequence. In order to give an intuitionistically correct, semantic, proof of the disjunction property and the explicit definability theorem, we define a condition $C(M_o, M, s)$ on M_o, M and s :

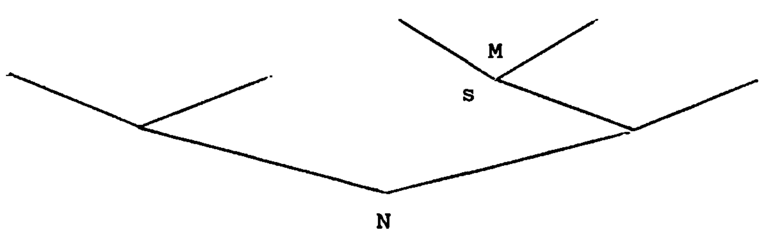
<u>Definition 1.5</u>: $C(M_o, M, s) \underset{D}{\equiv}$ for all m-ary predicate symbols P, for all m-tuples $\langle n_1, \ldots, n_m \rangle$ of natural numbers and for all $m \in \mathbb{N}$, if $T_{M_o}(s)(\langle P, n_1, \ldots, n_m \rangle) = 1$, then $T_M(\langle\,\rangle)(\langle P, n_1, \ldots, n_m \rangle) = 1$.
($\langle\,\rangle$ denotes the empty sequence).

If $C(M_o, M, s)$ holds, then we can built from M_o and M a new model $R(M_o, M, s)$ by replacing in the model M_o the submodel $M_o(s)$ by the model M ; more precisely:

<u>Definition 1.6</u>: Under the condition $C(M_o, M, s)$ let $R(M_o, M, s)$ by the model $N = \langle S, T_N \rangle$ defined by

$T_N(s')(\langle P, n_1, \ldots, n_m \rangle) = T_{M_o}(s')(\langle P, n_1, \ldots, n_m \rangle)$
if not s R s'

$T_N(s')(\langle P, n_1, \ldots, n_m \rangle) = T_M(s'')(\langle P, n_1, \ldots, n_m \rangle)$ if
$s' = s * s''$



In order to know that $N = R(M_o, M, s)$ is a model, we have to check that

if $T_N(<a_1, \dots, a_n>)(<P, n_1, \dots, n_m>) = 1$, then

$T_N(<a_1, \dots, a_{n+1}>)(<P, n_1, \dots, n_m>) = 1$, which holds because of the condition $C(M_o, M, s)$.

Lemma 1.7: Suppose $C(M_o, M, s)$ and let $N = R(M_o, M, s)$. Then for all α, for all formulas A and for all $a \in \mathbb{N}^\infty$

$$N_{s*\alpha} \models A[a] \quad \text{iff } M_\alpha \models A[a]$$

Proof: by induction on the complexity of A.

Theorem 1.8 (Disjunction property): Let A and B be sentences.

If $\models A \vee B$, then $\models A$ or $\models B$.

Proof: Suppose $\models A \vee B$, i.e. for all models $M \in W$ and for all α, $M_\alpha \models A$ or $M_\alpha \models B$.

Hence, for each $<M, \alpha> \in W \times \sigma_\omega$, there is $n \in \{0,1\}$, such that ($n=0$ and $M_\alpha \models A$) or ($n=1$ and $M_\alpha \models B$).

(σ_ω is the universal spread)

By the continuity-principle we can conclude:

For each $<M, \alpha> \in W \times \sigma_\omega$, there are $<k_1, k_2> \in \mathbb{N}^2$ and $n \in \mathbb{N}$, such that for all $<N, \beta> \in W \times \sigma_\omega$, if $\bar{N}k_1 = \bar{M}k_1$ and $\bar{\beta}k_2 = \bar{\alpha}k_2$, then ($n = 0$ and $N_\beta \models A$) or ($n=1$ and $N_\beta \models B$)

Now, let M_o be defined by $T_{M_o}(s)(<P, n_1, \dots, n_m>) = 0$ for all finite sequences s and for all $<P, n_1, \dots, n_m>$. And let $\underline{o}$ be defined by $\underline{o}(k) = 0$ for all $k \in \mathbb{N}$.

Then there are k_1 and k_2 and a number $n \in \{0,1\}$, such that for all $N \in W$ and for all $\beta \in \sigma_\omega$, if $\bar{N}k_1 = \bar{M}_o k_1$ and $\bar{\beta}k_2 = \bar{\underline{o}}\ k_2$, then ($n=0$ and $N_\beta \models A$) or ($n=1$ and $N_\beta \models B$)

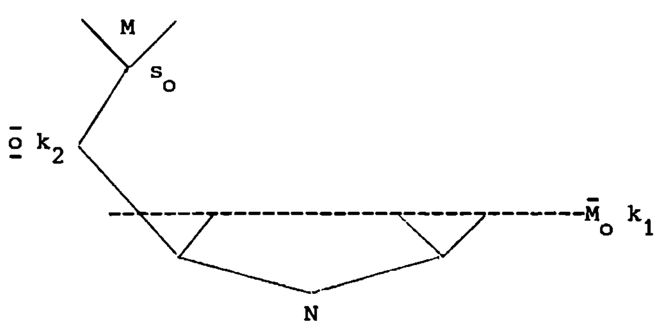
(*)

Looking at the coding of M_o as an infinite sequence of natural numbers in the proof of I, theorem 2.7, we see that in $\bar{M}_o k_1$ only information is given about finitely many nodes, say $s_1 , \ldots , s_p$.
For simplicity we can suppose that $k_2 \geq$ length (s_i) for all i , $1 \leq i \leq p$.

(**)

Proposition: if $n = 0$, then $\models A$
if $n = 1$, then $\models B$

Proof of proposition: Suppose $n = 0$ and let $M \in W$ be given. Let $s_o \underset{D}{\equiv} \bar{\underline{o}}\ k_2 * \langle 1 \rangle$.
Because $C(M_o , M, s_o)$ holds, we can construct the model $N = R(M_o , M , s_o)$.



Then, by (**) $\bar{N}\, k_1 = \bar{M}_o\, k_1$.

Hence, because n = 0, by (*), for all β, if $\bar{\beta}k_2 = \bar{\underline{o}}\, k_2$, then $N_\beta \models A$.

Hence, for all β through s_o , $N_\beta \models A$.

So, by lemma 1.7, for all β , $M_\beta \models A$; i.e. $M \models A$.

So we have shown that $M \models A$ for all $M \in W$, i.e. $\models A$, in case n = 0.

In case n = 1 we find in an analogous way $\models B$.

q.e.d.

In a way analogous to the proof of theorem 1.8, one proves

Theorem 1.9 (Explicit definability): Let (Ex)A(x) be a sentence.

If $\models$ (Ex)A(x), then there is $n \in \mathbb{N}$ such that $\models$ A (v) [n].

Lemma 1.10: Let (x)A(x) be a sentence.

$\models$ (x)A(x) iff for all $n \in \mathbb{N}$, $\models$ A(v) [n] .

Proof: $\models$ (x)A(x) (see I, definition 2.2 and 2.8)

$\Leftrightarrow$ for all M, for all α and for all $n \in \mathbb{N}$, $M_\alpha \models$ A(v)[n]

$\Leftrightarrow$ for all $n \in \mathbb{N}$, for all M and for all α, $M_\alpha \models$ A(v)[n]

$\Leftrightarrow$ for all $n \in \mathbb{N}$, $\models$ A(v) [n] .

q.e.d.

Modifying the proof of theorem 1.8 we get

Theorem 1.11: Let P and Q be propositional variables; A and B sentences.

i) If $\models (P\to Q) \to A \vee B$, then $\models (P\to Q) \to A$ or $\models (P\to Q)\to B$.

ii) If $\models (P\&Q) \to A \vee B$, then $\models (P\&Q) \to A$ or $\models (P\&Q)\to B$.

<u>Remark</u>: This theorem will be useful in establishing independence-results for the connectives $\vee$, &, $\rightarrow$ and $-$ (§ 3).

For a more general formulation of a syntactic analogue of this theorem see S.C. Kleene [24] .

<u>Proof</u>: i) Suppose $\models (P \rightarrow Q) \rightarrow A \vee B$, i.e. for all $M \in W$ and for all α, if $M_\alpha \models P \rightarrow Q$, then $M_\alpha \models A$ or $M_\alpha \models B$.

Now it is easy to see that the models M, such that if $T_M(s)(P) = 1$, then $T_M(s)(Q) = 1$ form a fan, say $W(P \rightarrow Q)$, which is a subfan of W. From $\models (P \rightarrow Q) \rightarrow A \vee B$ it follows that for all models $M \in W(P \rightarrow Q)$ and for all α , $M_\alpha \models A$ or $M_\alpha \models B$.

Using Brouwer's continuity principle we can conclude:
For each $\langle M,\alpha\rangle \in W(P \rightarrow Q) \times \sigma_\omega$, there are $\langle k_1, k_2\rangle \in \mathbb{N}^2$ and $n \in \mathbb{N}$ such that for all $\langle N,\beta\rangle \in W(P \rightarrow Q) \times \sigma_\omega$ if $\bar{N}k_1 = \bar{M}k_1$ and $\bar{\beta}k_2 = \bar{\alpha}k_2$, then ($n = 0$ and $N_\beta \models A$) or ($n = 1$ and $N_\beta \models B$).

Now, let M_o be defined by $T_{M_o}(s)(R) = 0$ for all propositional variables R and for all finite sequences s. And let $\underline{o}$ be defined by $\underline{o}(k) = 0$ for all $k \in \mathbb{N}$.

Clearly $M_o \in W(P \rightarrow Q)$ and hence there are k_1 and k_2 and a number $n \in \{0,1\}$, such that for all $N \in W(P \rightarrow Q)$ and for all $\beta \in \sigma_\omega$, if $\bar{N}k_1 = \bar{M}_o k_1$ and $\bar{\beta}k_2 = \bar{\underline{o}}\, k_2$, then ($n = o$ and $N_\beta \models A$) or ($n = 1$ and $N_\beta \models B$). (*)

Looking at the coding of M_o as an infinite sequence of natural numbers in the proof of I, theorem 2.7, we see that in $\bar{M}_o k_1$ only information is given about finitely many nodes, say $s_1, \ldots, s_p$.

For simplicity we can suppose that $k_2 \geqslant$ length (s_i) for all $i \leqslant p$. (**)

Proposition: if $n = 0$, then $\models (P \to Q) \to A$

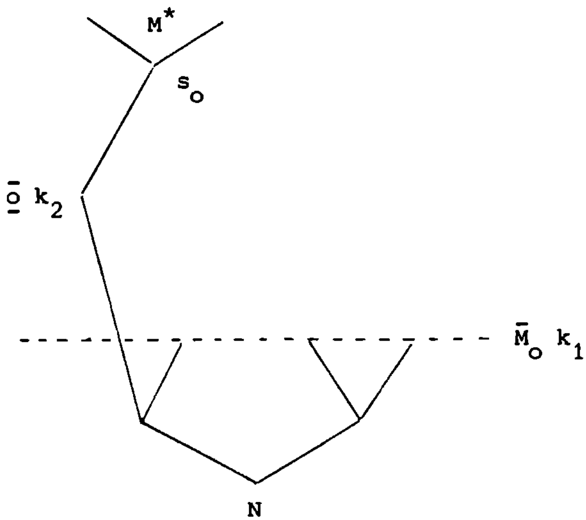
if $n = 1$, then $\models (P \to Q) \to B$

Proof of proposition: Suppose $n = 0$. Let $M \in W$, $\alpha \in \sigma_\omega$ and suppose $M_\alpha \models P \to Q$.
Then there is some $k \in \mathbb{N}$ (lemma I, 1.4) such that for all β, if $\bar{\beta}k = \bar{\alpha}k$, then $M_\beta \models P \to Q$.
Now consider the submodel $M(\bar{\alpha}k)$ (see definition 1.2).
By lemma 1.3, $M(\bar{\alpha}k) \models P \to Q$.

Because not necessarily $M(\bar{\alpha}k) \in W(P \to Q)$, we construct a model M^* such that
(1) $M^* \in W(P \to Q)$ and
(2) $M(\bar{\alpha}k)_\beta \models C$ iff $M^*_\beta \models C$ for all β and for all formulas C.
Namely, define $M^* = \langle S, T_{M^*}\rangle$ by $T_{M^*}(s)(R) = T_{M(\bar{\alpha}k)}(s)(R)$ if $R \neq Q$ and

$$T_{M^*}(s)(Q) = 1 \quad \text{if } T_{M(\bar{\alpha}k)}(s)(Q) = 1 \text{ or } T_{M(\bar{\alpha}k)}(s)(P) = 1$$
$$= 0 \text{ otherwise}$$

Let $s_o \underset{D}{=} \bar{o}\, k_2 * \langle 1\rangle$.
Because the condition $C(M_o, M^*, s_o)$ is satisfied, we can construct the model $N = R(M_o, M^*, s_o)$.



Then, by (**), $\bar{N}\ k_1 = \bar{M}_o k_1$. Also $N \in W\ (P \to Q)$.

So, because $n = 0$, by (*), for all β , if $\bar{\beta}k_2 = \bar{o}\ k_2$ then $N_\beta \models A$.

Hence for all β through s_o , $N_\beta \models A$.

Hence, by lemma 1.7 , $M^*{}_\beta \models A$ for all β .

So, by (2) , $M(\bar{\alpha}k)_\beta \models A$ for all β.

Then, by lemma 1.3, $M_{\bar{\alpha}k*\beta} \models A$ for all β.

So, in particular, $M_\alpha \models A$.

And $\models (P \to Q) \to A$, in case $n = 0$, has been proved.

In case $n = 1$ we find in an analogous way, $\models (P \to Q) \to B$.

q.e.d.

ii) The proof of ii) is analogous to the proof of i) with two modifications :
Instead of W $(P \rightarrow Q)$ consider now the fan W $(P \,\&\, Q)$ of all models M such that $T_M(<\ >)(P) = T_M(<\ >)(Q) = 1$. And define M_o, by $T_{M_o}(s)(P) = T_{M_o}(s)(Q) = 1$ for all finite sequences s and $T_{M_o}(s)(<R, n_1, \ldots, n_m>) = 0$ for all $R \neq P$, $R \neq Q$ and for all s.

q.e.d.

Remark: Let (*) denote: if $\vdash C \rightarrow A \vee B$, then $\vdash C \rightarrow A$ or $\vdash C \rightarrow B$.

In [23] R. Harrop proved that, under a condition RH on C, (*) holds in the intuitionistic propositional calculus.
In [24] S.C. Kleene shows that, under a weaker but not effective restriction $C \mid C$ on C than Harrop's R H, (*) holds in the intuitionistic propositional and predicate calculus.

One might hope to be able to prove the semantic analogue of (*) by a method, analogous to the proof of theorem 1.8 and 1.11.
As in remark 3.9 of chapter II, one can construct a fan of models, such that for each model M in the fan, $M \models' C$ (for the definition of $M \models' C$ see I, 2.12).
However, the analogue of lemma 1.7 with $\models'$ instead of $\models$:

If $N = R(M_o, M, s)$, then $N_{s*\alpha} \models' A \Leftrightarrow M_\alpha \models' A$

does not hold, because it may happen that N explodes, while M does not explode.

For cut-free systems there is a proof of the property (*), under Harrop's condition R H on C , in [27, page 55] and in [29 , page 336] .

§2 The model $_{i}\Sigma_{\epsilon IN}\, M_i$; Uniformity Properties; Compactness theorems.

Given an infinite sequence of models M_i , $i \in IN$, we can construct a new model $_{i}\Sigma_{\epsilon IN}\, M_i$ as follows:

<u>Definition 2.1</u>: $_{i}\Sigma_{\epsilon IN}\, M_i$ is the model $M = \langle S, T_M \rangle$ defined by:

$T_M\ (\bar{\underline{o}}\ k)\ (\langle P, n_1, \ldots, n_m \rangle) = 0$ for all $k \in IN$, for all m-ary predicate symbols P and for all m-tuples $\langle n_1, \ldots, n_m \rangle$ of natural numbers ($\underline{o}$ is defined by $\underline{o}\ (k) = 0$ for all $k \in IN$);

and for all $i \geqslant 1$ (see the picture)

$T_M\ (\langle i, a_1, \ldots, a_k \rangle) = T_{M_o}\ (\langle a_1, \ldots, a_k \rangle)$

$T_M\ (\langle o, i, a_1, \ldots, a_k \rangle) = T_{M_1}\ (\langle a_1, \ldots, a_k \rangle)$

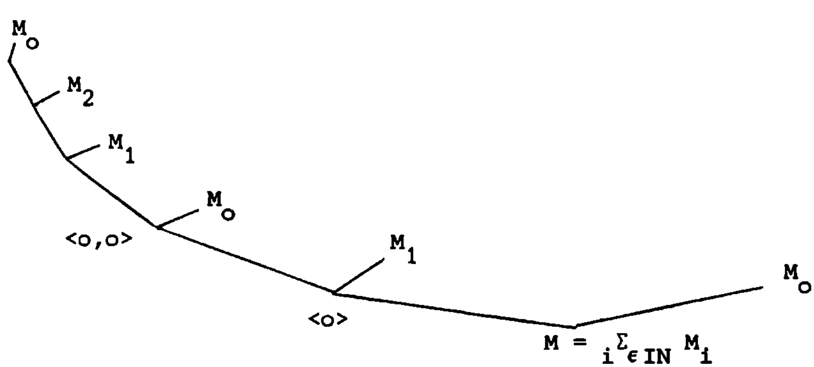
$T_M\ (\langle o, o, i, a_1, \ldots, a_k \rangle) = T_{M_o}\ (\langle a_1, \ldots, a_k \rangle)$

$T_M\ (\langle o, o, o, i, a_1, \ldots, a_k \rangle) = T_{M_1}\ (\langle a_1, \ldots, a_k \rangle)$

$T_M\ (\langle o, o, o, o, i, a_1, \ldots, a_k \rangle) = T_{M_2}\ (\langle a_1, \ldots, a_k \rangle)$

$T_M\ (\langle o, o, o, o, o, i, a_1, \ldots, a_k \rangle) = T_{M_o}\ (\langle a_1, \ldots, a_k \rangle)$

and so on. (We consider successively M_o , M_1 ; M_o , M_1 , M_2 ; M_o , M_1 , M_2 , M_3 ; M_o , M_1 , M_2 , M_3 , M_4 ; and so on)



<u>Lemma 2.2</u>: Let $M = {}_{i}\Sigma_{\epsilon\ IN}\ M_i$ and let $\underline{o}$ be defined by

$\underline{o}\ (k) = 0$ for all $k \epsilon$ IN.

For all formulas A and for all $a \epsilon IN^{\infty}$:

if $M_{\underline{o}} \models A[a]$, then $M_i \models A[a]$ for all $i \epsilon$ IN.

Proof: Suppose $M_{\underline{o}} \models A[a]$ and let $i \epsilon$ IN.

By lemma 2.4 of chapter I, there is some $k \epsilon$ IN such that for all β, if $\bar{\beta}k = \bar{\underline{o}}k$, then $M_\beta \models A[a]$. (1)

Take $l \geq k$ such that $T_M (\bar{\underline{o}}\ l * \langle i \rangle) = T_{M_i} (\langle\ \rangle)$.

By induction on the complexity of a formula A one shows that for all β

$$(M_i)_\beta \models A[a] \Leftrightarrow M_{\bar{\underline{o}}\ l * \langle i \rangle * \beta} \models A\ [a] \qquad (2)$$

From (1) and (2) it follows that for all β,

$(M_i)_\beta \models A[a]$, i.e. $M_i \models A\ [a]$.

q.e.d.

We have seen in theorem I 3.2, that there is a model M such that $M \models A$ iff $\models A$. For the case of the propositional calculus we will give now another proof of this theorem.

<u>Theorem 2.3</u>: For the propositional calculus there is a model M, such that for each sentence A, $M \models A$ iff $\models A$.

Proof: Let $A_1, A_2, A_3, \ldots$ be an enumeration of the formulas of the propositional calculus. Then, as remarked in chapter I after theorem 3.7, for each $i \epsilon$ IN it is decidable whether $\models A_i$ or for some $M \epsilon W$ not $M \models A_i$ (W is the fan of all models, see I, theorem 2.7).

Hence, for each $i \in IN$ there is some model $M \in W$, such that $\models A_i$ or not $M \models A_i$. Using the countable axiom of choice, which is intuitionistically acceptable, we find models M_1, M_2, M_3,... such that for each $i \in IN$,

$\models A_i$ or not $M_i \models A_i$. (*)

Now, let $M \underset{D}{=} {}_i\Sigma_{\in IN}\, M_i$ (see definition 2.1).

Proposition : for each formula A, $M \models A$ iff $\models A$.

Proof of proposition: From right to left by the definitions. Conversely: Suppose $A = A_i$ and $M \models A_i$. Then $M_{\underline{o}} \models A_i$.
Hence by lemma 2.2 $M_i \models A_i$.
Hence, by (*) , $\models A_i$. q.e.d.

In theorem 2.4 and 2.5 below, we will prove some <u>uniformity properties</u>.

<u>Theorem 2.4</u>: Let Γ be a countably infinite sequence of sentences.

i) If for all $A \in \Gamma$ there is a model $M \in W$ such that not $M \models A$, then there is a model $M \in W$ and an admissible α, such that for all $A \in \Gamma$ not $M_\alpha \models A$.

ii) If for all $A \in \Gamma$ there is a model $M \in W$ such that if $M \models A$, then M explodes, then there is a model $M \in W$ and an admissible α, such that for all $A \in \Gamma$, if $M_\alpha \models A$, then M explodes.

Proof: Analogous to the proof of theorem 2.3.

In theorem 2.4 we have seen, that, for Γ a countably infinite sequence of sentences, we can interchange $(A)_{A \in \Gamma}$ $(EM)_{M \in W}$ and $(EM)_{M \in W}$ $(A)_{A \in \Gamma}$ if the predicate is of the form "not $M \models A$" or of the form "if $M \models A$, then M explodes".

In corollary 3.4 of chapter I we have proved the following uniformity result: if $(M)_{M \in W}$ $(EA)_{A \in \Gamma}[M \models A]$, then $(EA)_{A \in \Gamma}$ $[\models A]$.
The proof made use of theorem I, 3.2, in which one model M is given such that, if $M \models A$, then $\vdash_{\overline{IPC}} A$.
Making use only of Brouwer's Continuity Principle and of purely semantic methods, we can prove a uniformity result, which is a bit stronger:

Theorem 2.5 : Let Γ be a countably infinite sequence of sentences.

i) If for all $M \in W$ and for all α there is some $A \in \Gamma$ such that $M_\alpha \models A$, then there is some $A \in \Gamma$ such that $\models A$, i.e. such that for all $M \in W$ and for all α, $M_\alpha \models A$.

ii) If for all $M \in W$ and for all α there is some $A \in \Gamma$ such that if $M_\alpha \models A$, then $M_\alpha \models f$, then there is some $A \in \Gamma$ such that $\models - A$, i.e. such that for all $M \in W$ and for all α, if $M_\alpha \models A$ then $M_\alpha \models f$.

Proof: analogous to the proof of theorem 1.8.

Notice that theorem 1.8 (Disjunction property) is a consequence of theorem 2.5 (i) :

Take $\Gamma = \{A, B\}$ and suppose $\models A \vee B$.
Then the hypothesis of theorem 2.5 (i) is satisfied and its conclusion says in this case that $\models A$ or $\models B$.

In chapter I we have already shown two versions of the compactness theorem (corollary I, 3.3 and I, 3.8), making use of the proofs of the completeness theorem.

In theorem 2.6 and 2.7 below, we will prove some other versions of compactness, making use of Brouwer's Continuity Principle, the fan theorem and purely semantic methods.

Theorem 2.6: Let Γ be a countably infinite sequence of sentences.

i) If for all $M \in W$ there is some $A \in \Gamma$ such that $M \models - A$, then there is a sentence $A \in \Gamma$ such that $\models - A$.

ii) If for all $M \in W$ there is some $A \in \Gamma$ such that, if $M \models A$, then M explodes, then there is some finite subset Γ' of Γ such that for all $M \in W$ there is some $A \in \Gamma'$ with the property, if $M \models A$, then M explodes.

Remark: In a classical translation, theorem 2.6 says: if Γ has no model, then there is some finite subset Γ' of Γ , which has no model.

Proof: i) analogous to the proof of theorem 1.8, using Brouwer's Continuity Principle and purely semantic methods. Another type of proof has already been given in corollary 3.4 of chapter I.

ii) Suppose for all $M \in W$ there is some $A \in \Gamma$ such that if $M \models A$, then M explodes.
By the continuity-principle we can conclude:
For all $M \in W$ there is some $k \in \mathbb{N}$ and some $A \in \Gamma$, such that for all $N \in W$, if $\bar{N}k = \bar{M}k$ and $N \models A$, then N explodes.

By the fan-theorem (W is a fan, see I, theorem 2.7) there is some $k \in \mathbb{N}$ such that for all $M \in W$ there is some $A \in \Gamma$ such that for all $N \in W$, if $\bar{N}k = \bar{M}k$ and $N \models A$, then N explodes.

Now, there are only finitely many different $\bar{M}k$ with $M \in W$, say $\bar{M}_1k, \ldots, \bar{M}_nk$.

Say, to M_1 we find $A_1, \ldots,$ to M_n we find A_n.
Take $\Gamma' = \{A_1, \ldots, A_n\}$.
Then for all $N \in W$ there is some $i \leq n$, such that if $N \models A_i$, then N explodes.

In other words: for all $M \in W$ there is some $A \in \Gamma'$ such that, if $M \models A$, then M explodes.

q.e.d.

<u>Theorem 2.7</u>: Let Γ be a countably infinite sequence of sentences.

i) If for each finite subset Γ' of Γ there is an $M \in W$ and an admissible α such that M does not explode and for all $A \in \Gamma'$ not $M_\alpha \models -A$, then there is a model

$M \in W$ and some α, such that M does not explode and for all $A \in \Gamma$ not $M_\alpha \models -A$.

ii) If for each finite subset Γ' of Γ there is a non-exploding $M \in W$, such that for all $A \in \Gamma'$ not $M \models -A$, then there is a non-exploding model $M \in W$ such that for all $A \in \Gamma$ not $M \models -A$.

Remark: In a classical translation this theorem says: if each finite subset of Γ has a model, then Γ has a model.

Proof: Let $A_1, A_2, A_3, \ldots$ be an enumeration of Γ.

The proof of i) is a specification of the proof of ii).

ii) Suppose for each finite subset Γ' of Γ there is a non-exploding model $M \in W$, such that for all $A \in \Gamma'$ not $M \models -A$.

Suppose, to $\{A_1\}$ we find M_1

to $\{A_1, A_2\}$ we find M_2

to $\{A_1, A_2, A_3\}$ we find M_3

and so on (countable axiom of choice, which is intuitionistically acceptable).

Let $M =_D \sum_{i \in IN} M_i$ (see definition 2.1)

Because for all $i \in IN$, M_i is non-exploding, M is non exploding.

Now suppose $A = A_i \in \Gamma$ and $M \models -A_i$.

Then by lemma 2.2 $M_i \models -A_i$.

Contradiction with not $M_i \models -A_i$.

Hence, for all $A \in \Gamma$ not $M \models -A$. q.e.d.

§3 The semantic independence of the intuitionistic connectives $\vee$, & ,$\rightarrow$ and $-$.

In [25] , McKinsey shows by a matrix-type proof that the four intuitionistic connectives $\vee$, &, $\rightarrow$ and $-$ are independent. See also [28] . It was not clear from his paper how the magic matrices were found. However, Krister Segerberg gives in [26] a translation of McKinsey's matrices in terms of Kripke models, defined on a finite set, which is partially ordered, but which does not have in all cases a tree structure.

Making use of the models, defined in chapter I, we will prove below, in an intuitionistically correct way, that

i) $\rightarrow$ is independent of $\vee$, & and $-$

ii) $\vee$ is independent of &, $\rightarrow$ and $-$

iii) $-$ is independent of $\rightarrow$,$\vee$ and &.

In [27] , pp. 59-62, Prawitz shows in a proof-theoretical way that the intuitionistic connectives are independent.

Let P, Q be propositional variables.
L (P, Q, $\vee$, &, $-$) is by definition the set of all formulas, constructed from P, Q, $\vee$, & and $-$, according to the usual rules. In an analogous way one defines L(P, Q, &,$\rightarrow$,$-$), L (P,$\rightarrow$,$\vee$, &) and L (P, Q, $\vee$, $\rightarrow$, $-$).

<u>Definition 3.1</u>: $\rightarrow$ is <u>independent</u> of $\vee$, & and $-$ $\underset{D}{=}$ for each formula A $\in$ L (P, Q, $\vee$, &, $-$) not

$$\models (P \rightarrow Q) \leftrightarrow A.$$

To prove that $\rightarrow$ is independent of $\vee$, & and $-$ we need

<u>Lemma 3.2</u> : If $\models (P \rightarrow Q) \rightarrow A$ and $A \in L(P, Q, \vee, \&, -)$,
then also $\models (P \rightarrow --Q) \rightarrow A$.

Proof:

$A = P$ or $A = Q$: There is nothing to prove.

$A = B \vee C$: Suppose $\models (P \rightarrow Q) \rightarrow B \vee C$.
Then by theorem 1.11 :
$\models (P \rightarrow Q) \rightarrow B$ or $\models (P \rightarrow Q) \rightarrow C$

Then, by induction hypothesis,
$\models (P \rightarrow --Q) \rightarrow B$ or $\models (P \rightarrow --Q) \rightarrow C$.

Hence $\models (P \rightarrow --Q) \rightarrow B \vee C$.

$A = B \& C$: Suppose $\models (P \rightarrow Q) \rightarrow B \& C$.

Then $\models (P \rightarrow Q) \rightarrow B$ and $\models (P \rightarrow Q) \rightarrow C$.

By induction hypothesis:
$\models (P \rightarrow --Q) \rightarrow B$ and $\models (P \rightarrow --Q) \rightarrow C$.

Hence $\models (P \rightarrow --Q) \rightarrow B \& C$.

$A = -B$: Suppose $\models (P \rightarrow Q) \rightarrow -B$.
Then $\models --(P \rightarrow Q) \rightarrow -B$.
Now, $\models (P \rightarrow --Q) \rightarrow --(P \rightarrow Q)$

Hence $\models (P \rightarrow --Q) \rightarrow -B$.

q.e.d.

From lemma 3.2 follows

<u>Theorem 3.3</u>: $\rightarrow$ is independent of $\vee$, & and $-$.

Proof: Suppose for some formula $A \in L(P, Q, \vee, \&, -)$

$\models (P \to Q) \leftrightarrow A$.

Then $\models (P \to Q) \to A$ and $\models A \to (P \to Q)$.

Then, by lemma 3.2, $\models (P \to --Q) \to A$ and $\models A \to (P \to Q)$

Hence $\models (P \to --Q) \to (P \to Q)$.

Contradiction.

q.e.d.

<u>Definition 3.4</u>: $\vee$ is independent of $\&$, $\to$ and $-$ $\overline{\overline{D}}$

for each formula $A \in L(P, Q, \&, \to, -)$

not $\models P \vee Q \leftrightarrow A$.

<u>Theorem 3.5</u>: $\vee$ is independent of $\&$, $\to$ and $-$.

Proof: We will give two different proofs:

i) Let $\square$ be one of the connectives $\vee$, $\&$, $\to$ and $-$.

Define: $\square$ is stable $\overline{\overline{D}}$ $\models --(A \square B) \leftrightarrow --A \square --B$.

Now, $\&$, $\to$ and $-$ are stable and each connective, composed from $\&$, $\to$ and $-$ is again stable, but $\vee$ is not stable. Hence $\vee$ is independent of $\&$, $\to$ and $-$.

ii) Let Γ be a set of formulas. By induction on the complexity of A we define $\Gamma \mid A$ as follows:

$\Gamma \mid P \;\overline{\overline{D}}\; \Gamma \vdash P$ (P propositional variable)

$\Gamma \mid B \,\&\, C \;\overline{\overline{D}}\; \Gamma \mid B$ and $\Gamma \mid C$

$\Gamma \mid B \vee C \;\overline{\overline{D}}\; (\Gamma \mid B$ and $\Gamma \vdash B)$ or $(\Gamma \mid C$ and $\Gamma \vdash C)$

$\Gamma \mid B \to C \;\overline{\overline{D}}$ if $(\Gamma \mid B$ and $\Gamma \vdash B)$, then $\Gamma \mid C$.

S.C. Kleene proves in corollary 2.7 of [24] , that if $C \mid C$ holds and $\vdash C \to D \vee E$, then $\vdash C \to D$ or $\vdash C \to E$ for the intuitionistic propositional calculus.

Now, suppose for some formula $A \in L(P, Q, \&, \rightarrow, -)$

$\models P \vee Q \leftrightarrow A.$ (1)

Then $\models P \rightarrow A$ and $\models Q \rightarrow A.$ (2)

Also $\models A \rightarrow P \vee Q.$

Because A does not contain $\vee$, $A \mid A$ holds ([24], 2.13) and hence, by Kleene's result and the soundness and completeness theorem of chapter I, section 3 it follows that $\models A \rightarrow P$ or $\models A \rightarrow Q$ (3)

By (2) and (3): $\models A \leftrightarrow P$ or $\models A \leftrightarrow Q$ (4)

Hence, by (1) and (4) : $\models P \vee Q \leftrightarrow P$ or $\models P \vee Q \leftrightarrow Q.$

Contradiction.

q.e.d.

<u>Definition 3.6</u>: $-$ is independent of $\&$, $\rightarrow$ and $\vee$ $\underset{D}{=}$ for each formula $A \in L(P, \&, \rightarrow, \vee)$ not $\models - P \leftrightarrow A$.

<u>Theorem 3.7</u>: $-$ is independent of $\&$, $\rightarrow$ and $\vee$.

Proof: Let $M = \langle S, T_M \rangle$ be defined by $T_M(s)(P) = 1$ and $T_M(s)(f) = T_M(s)(Q) = 0$ for all finite sequences s and for all propositional variables $Q \neq P$.

Then for each formula $A \in L(P, \&, \rightarrow, \vee)$ and for all α, $M_\alpha \models A$.

On the other hand for all α not $M_\alpha \models - P$.

Hence, there is no formula $A \in L(P, \&, \rightarrow, \vee)$ such that $\models - P \leftrightarrow A$.

q.e.d.

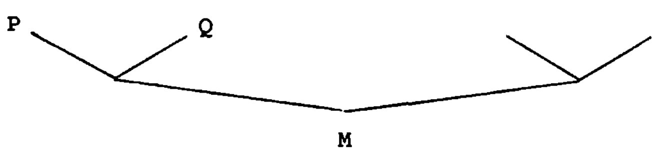
<u>Definition 3.8</u>: & is independent of $\vee$, $\rightarrow$ and $-$ $=_D$
for each formula $A \in L\,(P, Q, \vee, \rightarrow, -)$
not $\models P \,\&\, Q \leftrightarrow A$.

Unfortunately, I did not succeed in proving, only using the semantic concepts of chapter I, that & is independent of $\vee$, $\rightarrow$ and $-$.
I only have two partial results, stated in the following two lemma's:

<u>Lemma 3.9</u>: For each formula $A \in L\,(P, Q, \rightarrow, \vee)$
not $\models P \,\&\, Q \leftrightarrow A$, in other words:
& is not expressible in $\rightarrow$ and $\vee$.

Proof: Suppose for some $A \in L\,(P, Q, \rightarrow, \vee)$
$\models P \,\&\, Q \leftrightarrow A$.
Then for all $M \in W$ and for all admissible α
$M_\alpha \models P \,\&\, Q$ iff $M_\alpha \models A$. (1)

Now, consider the following model M:



Then for all α , not $M_\alpha \models P \,\&\, Q$. (2)

Proposition: For each $A \in L\,(P, Q, \rightarrow, \vee)$ there is some α such that $M_\alpha \models A$. (3)

Proof of proposition: $A = P$ or $A = Q$ is clear.

$A = B \vee C$: The result is immediate from the induction hypothesis .

$A = B \to C$: By induction hypothesis there is some α such that $M_\alpha \models C$.

By lemma I, 1.4 there is some $k \in \mathbb{N}$, such that for all β through $\bar{\alpha}k$, $M_\beta \models C$ and hence also if $M_\beta \models B$, then $M_\beta \models C$. So $M_\alpha \models B \to C$.

From (1), (2) and (3) follows a contradiction.

q.e.d.

Lemma 3.10 : For each formula $A \in L(P, Q, \vee, -)$ not $\models P \,\&\, Q \leftrightarrow A$, in other words: & is not expressible in $\vee$ and $-$.

Proof: by induction on the complexity of A.

$A = P$ or $A = Q$: trivial.

$A = B \vee C$: Suppose $\models P \,\&\, Q \leftrightarrow B \vee C$.

Then $\models B \to P \,\&\, Q$ and $\models C \to P \,\&\, Q$ (1)

By theorem 1.11 also: $\models P \,\&\, Q \to B$ or $\models P \,\&\, Q \to C$ (2)

From (1) and (2): $\models P \,\&\, Q \leftrightarrow B$ or $\models P \,\&\, Q \leftrightarrow C$, which contradicts the induction hypothesis.

$A = -B$: Suppose $\models P \,\&\, Q \leftrightarrow -B$.

Then also $\models --(P \,\&\, Q) \leftrightarrow -B$

Hence $\models --(P \,\&\, Q) \leftrightarrow P \,\&\, Q$.

Contradiction. q.e.d.

IV A FIRST STEP TOWARDS AN INTERPRETATION OF INTUITIONISTIC LOGIC IN INTUITIONISTIC ANALYSIS

If one asks an intuitionist to give a counterexample against P $\vee$ - P, he will probably answer that there is some spread σ_M and some predicate P_M , defined over the spread σ_M , such that not $(\alpha)_{\alpha \in \sigma_M} (P_M(\alpha) \vee - P_M(\alpha))$, for example $\sigma_M = \sigma_{01}$ and $P_M(\alpha) = (En)(\alpha(n) = 0)$.

This suggests the following concept of a model: a model M for the intuitionistic propositional calculus is a spread σ_M , together with a unary predicate P_M over the spread σ_M for each propositional variable P.

In [6] , page 100-105, S.A. Kripke shows that his notion of validity can readily be formulated in terms of Kreisel's theory FC of lawless sequences (absolutely free choice sequences).
In this chapter we will formulate our notion of validity from section 1 and 2 of chapter I in terms of intuitionistic analysis, using the full spread instead of the subspecies of all lawless sequences and without using axioms about lawless sequences.

§1 A first step towards an interpretation of intuitionistic propositional calculus in intuitionistic analysis.

Notation: For σ a spread, $SEQ(\sigma) \underset{D}{=} \{\bar{\alpha}k \mid \alpha \in \sigma \ \& \ k \in \mathbb{N}\}$.

Definition 1.1: A model $M = \langle \sigma_M , (P_i)_M \rangle_{i \in \mathbb{N}}$ for the intuitionistic propositional calculus is a spread σ_M, together with unary predicates $(P_i)_M$ ($i \in \mathbb{N}$) over the spread σ_M, such that for each $i \in \mathbb{N}$, $(P_i)_M : SEQ(\sigma_M) \to \{0,1\}$ and such that if $(P_i)_M (\bar{\alpha}k) = 1$, then $(P_i)_M (\bar{\alpha}k+1) = 1$.

Remark: The predicates over σ_M, associated with the functions $P_M : SEQ(\sigma_M) \to \{0,1\}$ are predicates of a very special type:
The intended interpretation of $P_M(\bar{\alpha}k) = 1$ is:
I know on the ground of $\bar{\alpha}k$ that α has the property P_M, more precisely, for all β, if $\bar{\beta}k = \bar{\alpha}k$, then β has the property P_M.
$M \models P\ [\alpha]$ will mean that $P_M(\bar{\alpha}k) = 1$ for some $k \in \mathbb{N}$, i.e. that there is an initial segment of α on the ground of which I know that α has the property P_M.
So we exclude for example the predicate $Q(\alpha) \underset{D}{=} (n)(\alpha(n) = 0)$: let $\underline{o}$ be defined by $\underline{o}(k) = o$ for all $k \in \mathbb{N}$; then $Q(\underline{o})$ holds, but we do not know $Q(\underline{o})$ on the ground of an initial segment $\bar{\underline{o}}k$ of $\underline{o}$.
This is the reason why we speak of "<u>a first step</u> towards an interpretation of intuitionistic logic in intuitionistic analysis".
There may be a whole hierarchy of predicates over a spread.

Note that in definition 1.2, below, f is a propositional symbol (absurdity), while $f_M : SEQ(\sigma_M) \to \{0,1\}$.

<u>Definition 1.2</u>: Let $M = \langle \sigma_M , (P_i)_M \rangle$ $i \in IN$ be a model and let $\alpha \in \sigma_M$.

$M \models f[\alpha] \underset{D}{\equiv} f_M(\bar{\alpha}k) = 1$ for some $k \in IN$.

$M \models A[\alpha] \underset{D}{\equiv}$

1. $M \models f[\alpha]$ or
2. $A = P$ and $P_M(\bar{\alpha}k) = 1$ for some $k \in IN$, or
3. $A = B \vee C$ and ($M \models B[\alpha]$ or $M \models C[\alpha]$), or
4. $A = B \,\&\, C$ and ($M \models B[\alpha]$ and $M \models C[\alpha]$), or
5. $A = B \to C$ and there is some $k \in IN$ such that for all β , if $\bar{\beta}k = \bar{\alpha}k$ and $M \models B[\beta]$, then $M \models C[\beta]$.

<u>Definition 1.3</u>: Let $M = \langle \sigma_M , (P_i)_M \rangle_{i \in IN}$ be a model.
$M \models A \underset{D}{\equiv} M \models A[\alpha]$ for all $\alpha \in \sigma_M$.

<u>Lemma 1.4</u>: $M \models A[\alpha]$ iff there is some $k \in IN$, such that for all β , if $\bar{\beta}k = \bar{\alpha}k$, then $M \models A[\beta]$.

Proof: by induction on the complexity of A.

<u>Remark</u>: Let A(P, Q) be a formula with P and Q as the only propositional symbols.

In chapter I a model M assigns to each propositional symbol P a basic sentence M(P) over the natural numbers and gives at the same time a possible development of our knowledge about those sentences.
$M_\alpha \models A(P, Q)$ means in chapter I that searching along the search path α I will find a proof of the sentence

$A(M(P), M(Q))$, which results from the formula $A(P, Q)$ by substituting for P and Q the sentences $M(P)$ and $M(Q)$ over the natural numbers.

In this chapter IV we have changed our terminology: <u>the search path α along which we search for a proof of $M(P)$, has been replaced by an element α of a spread which may have the property P_M on the ground of an initial segment of α.</u>

$M \models A(P, Q)$ $[\alpha]$ means in this chapter IV that there is an initial segment of α, on the ground of which I know that α has the property $A_M = A(P_M, Q_M)$, which results from the formula $A(P, Q)$ by substituting for the propositional symbols P and Q the predicates P_M and Q_M over σ_M.
And $M \models A(P, Q)$ then means that for all α, $M \models A(P,Q)$ $[\alpha]$, i.e. that for all $\alpha \in \sigma_M$ there is an initial segment of α, on the ground of which I know that α has the property $A_M = A(P_M, Q_M)$.

Because the notion of validity of this chapter IV results from the notion of validity of chapter I by just changing terminology, the notions of chapter I and IV are formally equivalent, as will be explicitly stated in theorem 2.2.

The purpose of this chapter IV is simply to show that the notion of validity of chapter I can be formulated in terms of intuitionistic analysis without admitting lawless sequences in our ontology and without using axioms about lawless sequences, which Kripke [6] has to use in order to be able to formulate his semantics in intuitionistic analysis.

Namely, given a model M, Kripke defines for each propositional variable P and for each lawless sequence α :

$P(\alpha) \underset{D}{\equiv}$ for some $k \in \mathbb{N}$, $M \models_{\bar{\alpha}k} P$

and

$(B \mathbin{\&} C)(\alpha) \underset{D}{\equiv} B(\alpha) \mathbin{\&} C(\alpha)$

$(B \vee C)(\alpha) \underset{D}{\equiv} B(\alpha) \vee C(\alpha)$

$(B \rightarrow C)(\alpha) \underset{D}{\equiv} B(\alpha) \rightarrow C(\alpha)$

In order to prove that, for a finite sequence s of natural numbers,

$M \models_{s} B \rightarrow C$ iff for all lawless sequences α through s ,
$B(\alpha) \rightarrow C(\alpha)$

Kripke needs the following axiom about lawless sequences α:

$B(\alpha)$ iff there is some $k \in \mathbb{N}$ such that for all lawless sequences β, if $\bar{\beta}k = \bar{\alpha}k$, then $B(\beta)$.

Because, in this paper, validity is defined in arbitrary infinitely proceeding sequences and not in nodes as in Kripke models and because of our definition of $M \models B \rightarrow C\,[\alpha]$, we were able to formulate our semantics directly in terms of intuitionistic analysis, with $M \models A\,[\alpha]$ instead of Kripke's $A(\alpha)$, using the full spread instead of the subspecies of all lawless sequences and without using axioms about lawless sequences.

Hence we do not have to admit lawless sequences in our ontology.

In Kripke's interpretation of [6] over all lawless sequences all predicates are admitted; in the interpretation over all choice sequences of this chapter we only admit predicates of a special type.

In Kripke's paper [6] '$(B \to C)(\alpha)$' is defined by '$B(\alpha) \to C(\alpha)$', α a lawless sequence, and it has the appropriate properties by the axiom, mentioned above, for lawless sequences.
In this chapter '$M \models B \to C\,[\alpha]$', α a choice sequence, has the appropriate properties by convention (definition 1.2).

§2 A first step towards an interpretation of intuitionistic predicate calculus in intuitionistic analysis.

Consider a language with predicate symbols P_i , $i \in \mathbb{N}$; suppose P_i is $\tau(i)$-ary, P_o is o-ary and let $f \underset{D}{\equiv} P_o$.

Definition 2.1: A model M =

$$< \sigma_M , (P_i)_M^{n_1 ,\ldots, n_{\tau(i)}} >_{i, n_1 ,\ldots, n_{\tau(i)} \in \mathbb{N}} \text{ is}$$

a spread σ_M , together with unary predicates $(P_i)_M^{n_1 ,\ldots, n_{\tau(i)}}$ $(i, n_1 ,\ldots, n_{\tau(i)} \in \mathbb{N})$ over the spread σ_M , such that for each $i, n_1 ,\ldots, n_{\tau(i)} \in \mathbb{N}$ $(P_i)_M^{n_1 ,\ldots, n_{\tau(i)}} : SEQ(\sigma_M) \to \{ 0,1\}$ and such that if $(P_i)_M^{n_1,\ldots, n_{\tau(i)}} (\bar{\alpha}k) = 1$, then $(P_i)_M^{n_1 ,\ldots, n_{\tau(i)}}$ $(\bar{\alpha}k+1) = 1$.

So, for example, a binary predicate symbol P is interpreted as a sequence $P_M^{o,o}$, $P_M^{o,1}$, $P_M^{1,o}$ $P_M^{1,1}$, $P_M^{2,0}$, $P_M^{0,2}$, $P_M^{2,1}$, ... of unary predicates over σ_M .

The intended meaning of $M \models P(v,w) [n, m,\alpha]$ is, that there is an initial segment of α , on the ground of which I know that α has the property $P_M^{n, m}$.

Let M be a model, $\alpha \in \sigma_M$, $A(v_1 ,\ldots, v_m)$ a formula and let $n_1 ,\ldots, n_m \in \mathbb{N}$.

$M \models A(v_1, \ldots, v_m)\ [n_1, \ldots, n_m, \alpha]$ is defined completely analogous to the definition of $M_\alpha \models A(v_1, \ldots, v_m)$ $[n_1, \ldots, n_m]$ in definition 2.2 of chapter I.

For P a binary predicate symbol, the intended meaning of $M \models (x)\ P(x, y)\ [m, \alpha]$ is that there is an initial segment of α, on the ground of which I know that for each $n \in \mathbb{N}$ α has the property $P_M^{n,m}$.

Analogous to definition 2.3 of chapter I we define:

$M \models A(v_1, \ldots, v_m)\ [n_1, \ldots, n_m] \underset{D}{\equiv}$

$M \models A(v_1, \ldots, v_m)\ [n_1, \ldots, n_m, \alpha]$ for all $\alpha \in \sigma_M$.

And by induction on the complexity of A one proves the analogue of lemma 2.4 of chapter I:

$M \models A(v_1, \ldots, v_m)\ [n_1, \ldots, n_m, \alpha]$ iff there is some $k \in \mathbb{N}$ such that for all $\beta \in \sigma_M$, if $\bar{\beta}k = \bar{\alpha}k$, then $M \models A(v_1, \ldots, v_m)\ [n_1, \ldots, n_m, \beta]$.

In theorem 2.2 below we state explicitly that the notions of validity of chapter I and of this chapter IV are formally equivalent. So, only the intended meaning is expressed in a different way.

<u>Theorem 2.2</u>:

i) Let $M = \langle S_M, T_M \rangle$ be a model in the sense of definition 2.1 of chapter I.
Define $N = \langle S_N, (P_i)_N^{n_1, \ldots, n_{\tau(i)}} \rangle_{i, n_1, \ldots, n_{\tau(i)} \in \mathbb{N}}$ as follows:
$S_N \underset{D}{\equiv} S_M$

$$(P_i)_N^{n_1, \ldots, n_{\tau(i)}} (\bar{\alpha}k) \underset{D}{\equiv} T_M (\bar{\alpha}k)(< P_i , n_1 , \ldots, n_{\tau(i)} >)$$

Then N is a model in the sense of definition 2.1 of this chapter IV and for all α

$N \models A(v_1 , \ldots, v_m) [n_1 , \ldots, n_m , \alpha]$ iff

$M_\alpha \models A(v_1 , \ldots, v_m) [n_1 , \ldots, n_m]$

ii) Let $N = < S_N , (P_i)_N^{n_1, \ldots, n_{\tau(i)}} >_{i, n_1 , \ldots, n_{\tau(i)} \in \mathbb{N}}$

be a model in the sense of definition 2.1 of this chapter IV.

Define $M = < S_M , T_M >$ as follows:

$S_M \underset{D}{\equiv} S_N$

$T_M (\bar{\alpha}k)(<P_i , n_1 , \ldots, n_{\tau(i)} >) = 1 \underset{D}{\equiv}$

$(P_i)_N^{n_1, \ldots, n_{\tau(i)}} (\bar{\alpha}k) = 1.$

Then M is a model in the sense of definition 2.1 of chapter I and for all α

$N \models A(v_1 , \ldots, v_m) [n_1 , \ldots, n_m, \alpha]$ iff

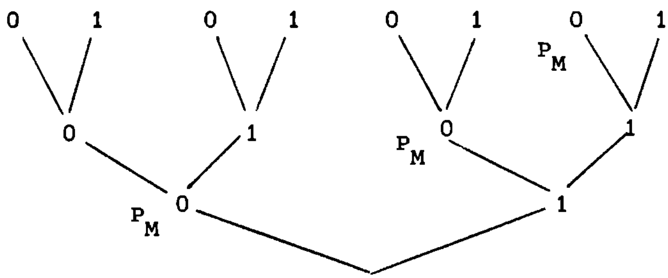
$M_\alpha \models A(v_1 , \ldots, v_m) [n_1 , \ldots, n_m]$.

Proof: immediate from the definitions.

§3 Discussion.

One can actually give two reasons for the invalidity of $P \vee - P$:

1. not $(\alpha)_{\alpha \in \sigma_{01}}$ ((En) (α n = o) $\vee$ - (En) (α n = o)), using Brouwer's continuity-principle.
2. not $M \models P \vee - P$, where $M = < \sigma_{01} , P_M >$ as defined in the following picture:



i.e. $P_M (\bar{\alpha}k) = 1$ iff $(El)_{1 \leq k}$ (α (l) = 0), hence

(Ek) ($P_M (\bar{\alpha}k) = 1$) iff (En) (α (n) = 0).

For α the right-most element of σ_{01} , i.e. $\alpha(k) = 1$ for all $k \in \mathbb{N}$, we have not $M \models P[\alpha]$ and not $M \models - P [\alpha]$, because if $M \models - P[\alpha]$, then by definition there was some $k \in \mathbb{N}$, such that for all β through $\bar{\alpha}k$ not $M \models P[\beta]$, which is not the case.

The first argument does, but the second argument does not make explicitly use of Brouwer's Continuity Principle. In the second argument we have put the effect of Brouwer's

Continuity Principle for an invidual choice-sequence into the definition of $M \models A[\alpha]$.

In an analogous way one can give two reasons for the invalidity of $-- P \rightarrow P$; these two reasons are less similar than the reasons, we gave for the invalidity of $P \vee - P$.

1. substituting $Q \vee - Q$ for P, $-- (Q \vee - Q) \rightarrow (Q \vee - Q)$ is not valid, using again Brouwer's continuity principle.

2. not $M \models -- P \rightarrow P$ with $M = \langle \sigma_{01} , P_M \rangle$ as defined before.

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N O T A T I O N S

S U B J E C T   I N D E X

S a m e n v a t t i n g

In de literatuur zijn diverse voorstellen gedaan voor een adaequate waarheidsdefinitie voor formules van de intuitionistische logica. In dit proefschrift wordt een nieuwe, intuitionistisch plausibele, waarheidsdefinitie geïntroduceerd, die diverse voordelen van technische aard blijkt op te leveren, vergeleken met de reeds bekende semantieken.

Terwijl men gewoonlijk de studie van de intuitionistische logica bedrijft in een klassieke metataal, wordt in dit proefschrift overal, zonder uitzondering, een intuitionistische metataal gehanteerd. Een aantal problemen die dit met zich meebrengt, zoals bijvoorbeeld het bewijzen van de disjunctie-eigenschap, zijn in dit proefschrift opgelost.

Het proefschrift bevat een intuitionistisch bewijs van de volledigheidsstelling en van diverse compactheidsstellingen, gebruik makende van een generalisatie van het begrip "model" voor een intuitionistisch systeem, zoals dit door drs. W.H.M. Veldman is ontwikkeld.

Curriculum Vitae

Ondergetekende, geboren op 5 september 1944 te Tilburg, behaalde aldaar aan het St. Odulphuslyceum het diploma Gymnasium β in 1962.

Hij studeerde wis- en natuurkunde aan de Katholieke Universiteit. 3 november 1966 legde hij cum laude het doctoraalexamen wiskunde af, met als specialisatie logica en grondslagen van de wiskunde.

Sinds die tijd was hij werkzaam op het Mathematisch Instituut te Nijmegen bij de afdeling grondslagen en wijsbegeerte van de wiskunde, tot 1 october 1971. Vanaf die datum is hij verbonden aan het Filosofisch Instituut te Nijmegen, waar hij sedert 1969 onderwijs geeft in de logica.

H. de Swart

STELLINGEN

I FILOSOFISCHE LOGICA

1 Evenals het nominalisme van N. Goodman, zoals dat in de onderstaande literatuur beschreven wordt, heeft ook het intuitionisme kritiek op het wetenschappelijk taalgebruik, met name met betrekking tot quantificatie over de collectie van alle verzamelingen. Echter, de beperkingen die N. Goodman oplegt aan het wetenschappelijk taalgebruik, blijken nauwelijks uitvoerbaar. Brouwer's quantificatie over de elementen van een spreiding blijkt daarentegen wel een practisch uitvoerbaar, partiëel, alternatief te bieden en geeft bovendien een veel diepere analyse van de betreffende problemen dan Goodman's nominalisme. Zie: N. Goodman and W. Quine, Steps towards a constructive nominalism. The Journal of Symbolic Logic, vol. 12, 1947, pp. 105-22.
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H. Putnam, Philosophy of logic.

2 N. Goodman's nominalisme heeft problemen met de quantificatie die voorkomt in de formulering van 'A is intuitief geldig'. Echter, hoewel in 'A is intuitief geldig' een quantificatie voorkomt, welke geen quantificatie over de elementen van een spreiding is, begrijpt een intuitionist naar mijn mening wat hij bedoelt met 'als A afleidbaar is, dan is A intuitief geldig', omdat de genoemde quantificatie voorkomt in het succedens van de implicatie. Maar om de betekenis van de omgekeerde bewering 'als A geldig is, dan is A afleidbaar' te kunnen begrijpen, moet een intuitionist zich beperken tot een quantificatie over de elementen van een spreiding, zoals in feite gedaan wordt in het begrip '$\models A$' van dit proefschrift.
Zie de literatuur, vermeld bij 1.

3 Het contrast tussen 'platonisme' en intuitionisme komt in wezen reeds voor in het commentaar van Proclos (± 450 na Christus) op het eerste boek van de Elementen van Euclides, waarin onderscheid wordt gemaakt tussen Speusippos en de zijnen (± 350 voor Christus), volgens welke alle constructie-opgaven theorema's zijn en Menaichmos en de mensen om hem heen (± 350 voor Christus), volgens welke alle theorema's constructie-opgaven zijn. Zie Paul Ver Eecke, Proclus de Lycie, les commentaires sur le premier livre des éléments d'Euclide, blz. 69-70.

II INTUITIONISTISCHE ANALYSE

4 Een volledige, totaal beperkte, metrische ruimte kan worden gerepresenteerd door een waaier. Dientengevolge geldt voor een dergelijke ruimte (E, ρ):

i) Iedere $f : E \to \mathbb{R}$ is uniform continu op E.

ii) Zij voor iedere $n \in \mathbb{N}$, $f_n : E \to \mathbb{R}$.
Als er voor iedere $x \in E$ een $y \in \mathbb{R}$ is, zódanig dat $\lim_{n\to\infty} f_n(x) = y$, dan is er een functie $g : E \to \mathbb{R}$ zódanig dat $\lim_{n\to\infty} \operatorname{unif}_{E} f_n = g$.

5 Voor (E, ρ) een totaal beperkte, metrische ruimte zij $C(E)$ de verzameling van alle uniform continue functies $f : E \to \mathbb{R}$.
$C(E)$ kan worden gerepresenteerd door een spreiding, welke geen waaier is en $(C(E), d)$, met d gedefinieerd door $d(f,g) = \|f-g\|$, is een volledige metrische ruimte.

6 Stelling van Stone-Weierstrass: zij (E, ρ) een volledige, totaal beperkte, metrische ruimte. Als A een deel-algebra is van $C(E)$, die de constante functies bevat en die de punten van E scheidt, dan is A dicht in $C(E)$.
N.B. A scheidt de punten van E $\underset{D}{=}$ voor alle $x, y \in E$, als $x \# y$, dan is er een $f \in A$ zódanig dat $f(x) \# f(y)$.
Gevolg: Als (E, ρ) een volledige, totaal beperkte, metrische ruimte is, dan is $(C(E), d)$ separabel.

III Hilary Putnam's boekje 'Philosophy of Logic' zou aanzienlijk grotere diepgang gehad hebben, indien hij ook het intuitionisme in zijn beschouwingen had betrokken.

H. de Swart Nijmegen, 2 april 1976